Basins of Attraction of the Two-Dimensional “Poor Man’s Navier–Stokes Equation”

S. A. Bible
Department of Mechanical Engineering
University of Kentucky, Lexington, KY 40506-0108

J. M. McDonough
Departments of Mechanical Engineering and Mathematics
University of Kentucky, Lexington, KY 40506-0108

ABSTRACT

This research is part of an ongoing effort to construct “synthetic velocity” subgrid-scale (SGS) models using discrete dynamical systems (DDSs) for use in large-eddy simulations of turbulent flows. Here we will outline the derivation of the two-dimensional (2-D) “Poor Man’s Navier–Stokes” (PMNS) equation from the 2-D, incompressible Navier–Stokes equation to be used in such models and report results from subsequent numerical investigations. In our results emphasis is placed on the understanding of the sensitivity to initial conditions inherent in the 2-D PMNS equation. Using system parameters we delineate fourteen types of behavior associated with this DDS and show examples of each. This study is a precursor to those already underway in curve fitting system (bifurcation) parameter values of the 3-D PMNS to relevant physical parameters (e.g. velocity gradients) and special consideration will be given to the consequences of this study on our ability to discern applicable ranges of bifurcation parameter values for the analogous 3-D PMNS.

Title: 2-D PMNS Equation
1. Introduction

Large-eddy simulation (LES) is one of the many prospective ways to model turbulent flows. Traditional LES entails performing direct simulation of the Navier–Stokes equations on grids that are sufficiently coarse for feasible calculations while attempting to model the subgrid-scale (SGS) analog of the Reynolds stress using largely heuristic SGS models (see Germano et al. [1991]). In the past Hylin and McDonough [1999] have pointed out that modeling turbulence in this fashion actually corresponds to modeling a first-order advection interaction between subgrid-scale and large-scale flow with a second-order diffusive operator. They propose, instead, to model the small-scale contributions to the dependent variables in a general turbulence modeling procedure they term additive turbulent decomposition (ATD) that was introduced considerably earlier by McDonough and Bywater [1986, 1989]. In the current version of ATD the small-scale velocity field is represented by:

\[ q^* = A \zeta M, \]

where \( q^* \) can be viewed as the small-scale portion of the usual LES decomposition of dependent variables,

\[ Q(x, t) = q(x, t) + q^*(x, t), \quad x \in \mathbb{R}^d, \quad d = 2, 3, \]

where \( q(x, t) \) denotes the large- or resolved-scale part. In Eq. (1) \( A \) is an amplitude factor derived from Kolmogorov scalings (see, e.g., Frisch [1995]); \( \zeta \) is an anisotropy correction computed via the scale-similarity hypothesis employed in dynamic SGS models [Germano, 1991], and \( M \) is the “stochastic variable” that introduces the turbulent-like fluctuations.

Recently ATD has been further enhanced by the direct derivation of the normalized stochastic variable, \( M \), from the 2-D, incompressible N.–S. equation [McDonough and Huang, 2001]. This “poor man’s N.–S. equation”, as Frisch [1995] termed similar quadratic maps, is a system of discrete dynamical equations derived using a Galerkin procedure on the Fourier expanded, incompressible N.–S. equation and might be viewed as the simplest possible “shell model” (see Bohr et al. [1998]) of the N.–S. equation. The 2-D PMNS equation is given here:

\[ a^{n+1} = a^{(n)} \beta_a \left( 1 - a^{(n)} \right) - \gamma_a a^{(n)} b^{(n)}, \]

\[ b^{n+1} = b^{(n)} \beta_b \left( 1 - b^{(n)} \right) - \gamma_b a^{(n)} b^{(n)}. \]

Differentiable dynamical systems and discrete dynamical systems, such as Eqs. (3), have been studied for over a century [Poincaré, 1897] leading to many significant findings. For information on this subject we refer the reader to the review by Ruelle [1987], and to the text of Alligood et al. [1996]. It is well known that DDSs such as Eqs. (3) are capable of producing chaotic time series [May, 1976(a)]. One hypothesis, proposed by Ruelle and Takens [1971] among others, is that turbulence is a chaotic phenomenon (rather than a random one) and is a consequence of a “strange attractor” of the N.–S. equation. Both numerical and physical experiments, including those of Gollub and Benson [1980], have provided support for this theory of strange attractors as opposed to the former Landau-Hopf hypothesis of a toroidal attractor corresponding to an ever increasing number of characteristic frequencies [Landau and Lifshitz, 1959].

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The goal of our current work is to infuse chaotic dynamics into models of the subgrid-scale fluctuations of the velocity using the 2-D PMNS equation to produce spatially and temporally accurate 2-D turbulence simulations (note though that questions concerning the realizability of 2-D turbulence enhance limitations of 2-D simulations). If the 2-D PMNS equation is to be used in LES or ATD as proposed, the basins of attraction of the aforementioned DDS must be delineated. Only then can we choose suitable ranges of parameter values to be used in the process of curve fitting experimental data so that relationships between the DDS bifurcation parameters and physical variables (e.g., velocity gradients) can be determined. A basin of attraction, loosely defined, is a closed set of initial conditions that produces similar long-term behavior for given values of bifurcation parameters [Alligood et al., 1996]. If we wish to delineate these basins we must use computationally efficient properties that can consistently distinguish between the many possible types of long term behavior. Here we investigate various DDS properties, statistical quantities, and turbulence parameters for use in this capacity and implement those found to be most successful as indicators of basins of attraction of the mappings, Eqs. (3), for selectively chosen sets of bifurcation parameters.

2. Mathematical Formulation

Here we will summarize the derivation of the 2-D PMNS equation, given in more detail by McDonough and Huang [2001]. We begin with the 2-D, incompressible N.-S. equation projected into a divergence-free Sobolev space;

\[ u_t + uu_x + vu_y = \frac{1}{Re} \Delta u, \]  
\[ v_t + uv_x + vv_y = \frac{1}{Re} \Delta v, \]  

along with the Fourier expanded velocity representations;

\[ u(x, t) = \sum_{k=\infty}^{\infty} a_k(t) \phi_k(x), \]  
\[ v(x, t) = \sum_{k=\infty}^{\infty} b_k(t) \phi_k(x). \]  

We then assume the basis functions, \( \phi_k \), are in the space \( C_0^\infty \), that they are divergence-free and orthogonal, and that they “act like” complex exponentials with respect to differentiation. By applying a Galerkin procedure and the orthogonality constraint while retaining only one wave vector it is found that the equations above simplify to:

\[ \dot{a} = -A^{(1)} a^2 - A^{(2)} ab - \frac{C}{Re} |k|^2 a, \]  
\[ \dot{b} = -B^{(1)} b^2 - B^{(2)} ab - \frac{C}{Re} |k|^2 b. \]  

In Eqs. (6) we have suppressed the \( k \) subscript notation; the \( A^i \) and \( B^i, i = 1,2 \), are associated with Galerkin triple products (see McDonough and Huang [2001]), and C is a normalization
constant. Using a forward Euler explicit time step integration, and the algebraic substitution,
\[
\frac{1 - C\tau |\mathbf{k}|^2/Re}{\tau A^{(4)}} = 1
\]
yields Eqs. (3).

A few aspects of this derivation are of particular importance. Foremost is that Eqs. (3) are not a unique result of Eqs. (4), but only one of infinitely many choices, particularly that corresponding to the retention of only one wave vector, assumed to be in a range sufficiently high to be of use in subgrid-scale modeling. Caution must be exercised when considering the universality of solutions to Eqs. (3) because it is known that time discretization causes errors in the solutions of differential equations, as noted by Bigge and Bohl [1985]. Furthermore, relationships between the bifurcation parameters \((\beta_u, \beta_v, \gamma_u, \gamma_v)\) and physics are not straightforward and are discussed by McDonough and Huang [2001]. Here we will restrict our study to the subspace of parameter values where \(\beta_u = \beta_v = \beta\) and \(\gamma_u = \gamma_v = \gamma\). These are not completely arbitrary choices, as \(\beta_u\) and \(\beta_v\) have the same relationship to the Reynolds number, but are primarily made for computational tractability. From this we can conclude that we may be omitting some physically realistic behavior, or possibly introducing physically erroneous behavior, by only considering Eqs. (3) with arbitrary bifurcation parameters. But, the focus here is specifically on the effects of initial conditions on the solutions of the 2-D PMNS equation, and these can be presented most clearly by implementing Eqs. (3) as we have done here.

To continue, if we are to delineate the basins of attraction of Eqs. (3) we must be able to accurately distinguish solution types in an automatic fashion using solution parameter values calculated during the iteration sequences. Any such parameter must be efficient to calculate and not require excessive computer storage because as many \(10^6\) trajectories may be considered during a single run. There are three main classes of solution parameters that we consider for this purpose: dynamical systems specific, general statistical, and turbulence specific parameters. The remainder of this section is dedicated to discussing these three types of parameters.

### 2.1 Dynamical systems specific parameters

Deterministic nonlinear systems such as Eqs. (3) have been proposed to describe a multitude of physical processes including turbulence in hydrodynamics ([Bohr et al., 1998], [Ruelle and Takens, 1971], and [Yorke et al., 1987]), concentrations of species in chemical reactions [Hoffman et al., 1987], and biological population dynamics [May, 1976(b)], and subsequent analysis of such systems has been well documented. DDSs produce as their defining output a time series as the iteration procedure progresses. As stated by Ruelle [1987], these time series usually exhibit transient behavior followed by any one of many types of stationary behavior, ranging in complexity from steady to chaotic, that correspond to a particular attractor, or subspace of \(\mathbb{R}^n\) on which trajectories of the iterations converge. In light of this, the goal of dynamical system analysis is most often to at least qualitatively, if not quantitatively, determine the geometric structure of the attractor being considered.

The most well known DDS parameter, the time series, as we will see, yields useful results
for manual inspection only when dealing with the most trivial behaviors, i.e., steady, periodic, low-frequency subharmonic, and divergent. Therefore we must rely on more sophisticated analytical tools in our solution identification procedure.

The most commonly referenced DDS parameters are the Lyapunov number and Lyapunov exponent. A defining characteristic of differentiable dynamical systems is sensitivity to initial conditions (SIC). The Lyapunov number provides a measure of the rate that two initially nearby trajectories separate as time evolves, and the Lyapunov exponent is the natural logarithm of the Lyapunov number (sometimes referred to as the characteristic exponent). Though they are fairly good indicators of types of behavior, the calculation of these parameters requires the evaluation of the Jacobian matrix of the system at each iteration and use of this to construct a product matrix. This process is numerically unstable and thus will not be used here.

An orbit that is nonperiodic and has a positive Lyapunov exponent is defined as a chaotic orbit. Along with SIC, chaotic orbits also exhibit what is known as a geometrically “fractal” attractor. Fractal geometries are loosely defined as those that have complex boundaries, often repeated structures (self-similarity), and a characteristic “size” measure (often referred to as an asymptotic or physical measure, or dimension) that is not an integer. In general, certain mappings are considered as useful in extracting information about this characteristic measure, e.g., Poincaré maps, time delay maps, and phase portraits ([Ruelle, 1987], [Alligood et al., 1996]). These types of mappings produce visually interesting and informative plots (and in the case of delay maps are part of a necessary step in reconstructing an attractor from partial data, as from laboratory experiments), but are useless in terms of automatic calculation of relevant parameters such as we are attempting here.

Though there are undoubtedly many other dynamical systems characterizations (especially correlation dimension and Kolmogorov entropy) which we are failing to consider, most are similar to those above in that they are computationally inefficient or require “babysitting” by the user during the calculation. Thus, we will not employ any dynamical system specific parameters for solution characterization in the present study.

### 2.2 General statistical parameters

A majority of all possible solution characterization parameters come to us from the field of statistical analysis (see McDonough et al. [1998] for treatment of these in a related context). The most common general statistical property is the time average of an observable of the system. Averages, along with the maxima and minima, are easily computed with minimal storage requirements, but provide little useful information about the structural appearance of the time series and cannot be used here. The statistical quantities variance and standard deviation provide more information than the average, yet neither is able to distinguish between a periodic trajectory and a steady trajectory. Thus these are also unsuitable for use in solution characterization in the present study.

A more useful parameter is the correlation of variables. In the case of our DDS we can calculate the cross correlation between the $u$ and $v$ velocity components as

$$\frac{\langle u', v' \rangle}{\|u'\|\|v'\|} \simeq \frac{\sum_{n=1}^{N} (u^{(n)} - \overline{u})(v^{(n)} - \overline{v})}{\left[\sum_{n=1}^{N} (u^{(n)} - \overline{u})^2\right]^{1/2} \left[\sum_{n=1}^{N} (v^{(n)} - \overline{v})^2\right]^{1/2}},$$

(8)
where \( \langle \cdot, \cdot \rangle \) denotes the usual \( L^2 \) inner product, \( \| \cdot \| \) is the corresponding norm, and overbars indicate the time average. Another important statistical property in the research of differentiable dynamical systems is the power spectral density (PSD). Here we have used a standard radix-2 fast Fourier transform (FFT) to construct the PSD to confirm our solution types, but not as an automatically-calculated solution characterization parameter (with the exception, as noted in Sec. 3, of producing overall regime maps).

### 2.3 Turbulence specific parameters

Two parameters that overlap categories are skewness and flatness. Both are statistical properties in that they characterize the “tail” of a distribution ([Bulmer, 1967], [Tennekes and Lumley, 1972]). We calculate the skewness of the velocity components as:

\[
S = \frac{\overline{u^3}}{\left( \overline{u^2} \right)^{3/2}} \simeq \frac{1}{N-1} \frac{1}{\overline{u^2}} \sum_{n=1}^{N} \left( u^{(n)} - \overline{u} \right)^3 \left( \overline{u^2} \right)^{-3/2}. \tag{9}
\]

The flatness, or kurtosis, of a velocity component we calculate with the equation

\[
F = \frac{\overline{u^4}}{\left( \overline{u^2} \right)^{2}} \simeq \frac{1}{N-1} \frac{1}{\overline{u^2}} \sum_{n=1}^{N} \left( u^{(n)} - \overline{u} \right)^4 \left( \overline{u^2} \right)^{-2}. \tag{10}
\]

Since these are often used in experimental turbulence studies we list them in this category and use both as solution characterization parameters.

Another important physical measurement that is often made is the average kinetic energy of the fluctuations. This variable is given by the formula:

\[
\overline{\epsilon} = \frac{1}{2} (u^2 + v^2) \simeq \frac{1}{2} \sum_{n=1}^{N} \left[ \left( u^{(n)} - \overline{u} \right)^2 + \left( v^{(n)} - \overline{v} \right)^2 \right]. \tag{11}
\]

The remaining parameter employed for solution characterization in this study is the second-order structure function. We note that the form we use is not the usual one involving the difference between values of a quantity at two spatial locations, but rather a form introduced by Jensen et al. [1991] specifically for studying shell models. For the \( p \)th-order structure function this takes the form

\[
S_p(n) = \langle |u_n|^p \rangle, \tag{12}
\]

where \( n \) is the shell index; and \( u_n \) is the dependent variable of the \( n \)th shell, and \( \langle \cdot \rangle \) now denotes the time average. The DDS being considered herein has a specific structure in that it consists of only a single shell \( (n = 1) \), but there are two variables associated with that shell. We have constructed structure functions based on only one of these using the formula

\[
S_2 = \langle a^2 \rangle. \tag{13}
\]

Thus, these are closely related to (but not identical with) the kinetic energy of Eq. (11). The
reader is no doubt aware of the general connection between kinetic energy and second-order structure functions and its use in construction of Kolmogorov scalings (again, see Frisch [1995]).

3. Results and Discussion

Here we will present results obtained from our numerical experiments on the DDS of Eqs. (3). The two main goals of this study are to determine the most efficient way of delineating the basins of attraction of Eqs. (3) and then to determine the basins of attraction for a small subset of the entire parameter space drawing conclusions concerning trends and structures with relative importance to curve fitting the 2-D PMNS equation’s bifurcation parameters to experimental data. Therefore, in light of the discussions of Sec. 2, we investigate the basins of attraction as determined by the $u^iv^j$ correlation, skewness, flatness, kinetic energy, and second-order structure function contours. This was accomplished by dividing the open unit square of continuous initial conditions (I.C.s) for Eqs. (3) into equally spaced I.C. grids measuring $[\epsilon, 1 - \epsilon] \times [\epsilon, 1 - \epsilon]$, $\epsilon > 0$ where $\delta a_0^0 = \delta b_0^0 = \delta u_0 = \delta v_0 = \epsilon$. In select cases we also vary the “grid spacing” using values of $\epsilon$ of 0.02, 0.01, 0.005, and 0.0025 to determine if the basin boundaries appear fractal. For all of the basins presented here the time series, power spectra and phase portraits were plotted to confirm the type of solution.

Most of the calculations reported were performed in series or parallel mode using double precision arithmetic on the HP Superdome symmetric multiprocessor at the University of Kentucky Computing Center. Additional computations were made on a HP J2240 workstation in the Computational Fluid Dynamics Laboratory at the University of Kentucky. The runs generally consisted of $5 \times 10^4$ iterations of Eqs. (3) for each set of initial conditions, $(a_0^0, b_0^0)$. The last $10^4$ iterations were used in the appropriate statistical analyses of Sec. 2. Power spectra were computed using the last 8192 points of the time series using a standard radix-2 FFT.

The method of producing regime maps (2-D bifurcation diagrams) of Eqs. (3) has been thoroughly discussed by McDonough and Huang [2001] and we will not treat this further except to note that an “image processing” algorithm described elsewhere [McDonough, 2002] was used to assign a numerical value to each type of behavior observed in the power spectra. Here we duplicate the procedure used by McDonough and Huang [2001] for four sets of initial conditions (one in each of the four quadrants of the unit square) to produce four analogous regime maps that will indicate the bifurcation parameter values to be chosen such that the basins of attraction are potentially “interesting.” These regime maps are shown in Figs. 1(a–d) along with the color table and corresponding solution types in Fig. 1(e).

We see from Fig. 1 that the 2-D PMNS equation produces thirteen different types of behavior, excluding divergence. This is a much larger range of behaviors than are usually attributed to the 1-D logistic map [Alligood et al., 1996]. We wish to investigate cases (sets of bifurcation parameter values) for which the basins of attraction will provide information concerning the SIC of each of these types of behavior and insight into choosing initial conditions during the curve fitting process mentioned previously. After much consideration we have chosen seven cases that encompass this philosophy. We remark that an indication of the effects of initial conditions are already suggested by Fig. 1. In particular, there are considerable differences in solution types found in subregions that are common from one part
Figure 1: Regime maps using the I.C.s; (a) $a^{(0)} = 0.98, b^{(0)} = 0.9$, (b) $a^{(0)} = 0.4, b^{(0)} = 0.8$, (c) $a^{(0)} = 0.2, b^{(0)} = 0.1$ and (d) $a^{(0)} = 0.51, b^{(0)} = 0.48$ along with (e) the color table.
of the figure to the next. It is reasonable to expect that this sensitivity to small changes in bifurcation parameter values will translate to SIC and, indeed, this will be demonstrated in the sequel.

3.1 Case 1 (periodic / divergent): $\beta = 3.3, \gamma = -0.3874$

The first case we present is the simplest in terms of solution types that we will examine and corresponds to Eqs. (3) with $\beta = 3.3, \gamma = -0.3874$. From Figs. 1(a–d) we see that for these values of bifurcation parameters the solutions are either in the periodic (red) basin or the divergent (black) basin for those four sets of initial conditions. The basins of attraction shown in Fig. 2 are delineated using kinetic energy contours and correspond to coordinate axes representing the initial conditions, $a^0$ and $b^0$, respectively. As can be seen, divergent and periodic trajectories are approximately equally likely with periodic trajectories concentrated more heavily along the diagonals of the unit square. We omit the power spectra and time series for this case as we assume that the reader is familiar with these two fundamental types of behavior.

![Figure 2: Basins of attraction for bifurcation parameter values of $\beta = 3.3, \gamma = -0.3874$ as determined by the kinetic energy. Grid refinement clockwise from top left: $50 \times 50$, $100 \times 100$, $200 \times 200$, $400 \times 400$](image)

The sequence of patterns in Fig. 2 is produced by refining the initial condition grid spacing as described in Sec. 2. The spacing between grid points is consecutively halved as we proceed in the clockwise direction. We start with the plot in the upper-left hand corner
using a grid spacing of 0.02 units and end in the lower-left hand corner using a grid spacing of 0.0025 units. From observations of the basins as grid refinement is performed we conclude that this particular basin map appears to display a fractal “topology” although we have not attempted a rigorous proof of this.

3.2 Case 2 (periodic / quasiperiodic): $\beta = 3.4, \gamma = -0.05$

From Figs. 1(a–d) we see that for bifurcation parameters values of $\beta = 3.4$ and $\gamma = -0.05$, there are at least two types of behavior that coexist. The I.C.s of Figs. 1(a) and (b) indicate quasiperiodic trajectories, and those from Figs. 1(c) and (d) indicate periodic trajectories. The basins of attraction for such periodic and quasiperiodic behaviors are shown in Fig. 3(a) and correspond to the values of the statistical property of skewness of the $u$-component of velocity. The four dark dots of Fig. 3(a) indicate the I.C.s used in constructing the regime maps of Figs. 1(a–d) and confirm the solution types found in those regimes maps. The checkerboard basin of Fig. 3(a) is an example of a basin geometry commonly seen in the context of relatively simple behavior and has also been observed for combinations of subharmonic and quasiperiodic trajectories, e.g., where $\beta_u = \beta_v = 3.57, \gamma_u = \gamma_v = -0.02$, and for in-phase and out-of-phase periodic trajectories in a similar study [Froyland, 1992].

Since there are an infinite number of points within each basin we can never test every point using time series and power spectral analyses to determine all possible behaviors. Instead, we rely on the consistency of the numerical values of the identification parameters of Sec. 2.3 and on testing of randomly selected points within each basin. We will discuss this in more depth later. Due to the extensive duration of one of the periods we can visibly see the quasiperiodic nature of the $u$-component velocity time series of an example trajectory (Fig. 3(b)). In many cases though quasiperiodicity is not as apparent and we must rely on the PSD for solution confirmation. The PSD for this case of quasiperiodic behavior is shown in Fig. 3(c).

![Figure 3](image.png)

Figure 3: (a) Skewness contours for $\beta = 3.4, \gamma = -0.05$, (b) quasiperiodic time series of $u$-component velocity, (c) quasiperiodic power spectra.
3.3 Case 3 (subharmonic / noisy subharmonic): $\beta = 3.55, \gamma = 0.32$

In case 3 we continue to vary $\beta$ and $\gamma$ so that we can show examples of the entire spectrum of behaviors associated with the 2-D PMNS equation. The types of behavior for the case where $\beta = 3.55$ and $\gamma = 0.32$ is uncertain for those I.C.s of Figs. 1(b–d), but for the I.C. of Fig. 1(a) we expect divergent trajectories. Other possible solution types appear to include subharmonic, noisy subharmonic, noisy quasiperiodic, and broadband w/ fundamental among others.

The basin identification map produced by the $uv$ correlation and shown in Fig. 4(a) is reasonable in that it displays two basins of attraction; one corresponding to subharmonic behavior and one corresponding to noisy subharmonic behavior. But, after inspection of the basins of attraction identified by the other parameters and subsequent power spectral analysis, it is found that this is not the correct basin mapping. This error is a natural consequence of the definition of the $u^\prime v^\prime$ correlation. The definition implies that when the $u$- and $v$-component velocities are attracted to the same finite frequency oscillator and are in-phase (as they are in this case) that the $u^\prime v^\prime$ correlation will have a value of positive one, the same as associated with steady trajectories. Also, in our computations we have

![Figure 4: Contours for $\beta = 3.55, \gamma = 0.32$ using (a) $u^\prime v^\prime$ correlation and (b) flatness along with (c) power spectra of $u$-component velocity of noisy subharmonic trajectories (blue-green) and subharmonic trajectories (light green) and (d) example time series of each.](image)

constructed the iteration sequence so that when it is apparent that a trajectory is divergent then the statistic measures are assigned specified values, equal to one in the case of the $u^\prime v^\prime$ correlation, allowing for faster computations. These are the reasons why the $u^\prime v^\prime$ correlation
does not distinguish the correct basin map, shown in Fig. 4(b), as determined by the flatness contours and containing a basin of divergence in the upper right-hand corner. We remark that inability of correlations to distinguish fundamentally different dynamics represents a basic flaw in any modeling technique relying on correlations, e.g., any form of Reynolds-Averaged Navier–Stokes (RANS) method.

In Fig. 4(c) we see the first example of a noisy power spectrum corresponding to a noisy subharmonic trajectory. The relationship between the noise shown in Fig. 4(c) and that observed in experiments has not, as yet, been established, and we propose that at least some experimental noise can be attributed to sources other than experimental error. The \( u \)-component velocity time series for this behavior is shown by the top plot of Fig. 4(d). The bottom plots of these figures correspond to the power spectrum and \( u \)-component velocity time series, respectively, for the subharmonic trajectories.

Case 3 displays the commonly observed quadrilateral region of divergent behavior (black basin of Fig. 4(b)). These regions usually occur at relatively high values of \( \beta \) and \( \gamma \). We can see this dependency in the regime map of Fig. 1(a). This is important because we will want to avoid using these initial conditions in the experimental ranges of curve fit parameters when attempting to fit behavior found in these regions of bifurcation parameter values. Moreover, subgrid-scale models employing these maps must be constructed to automatically avoid such regions. These regions of divergent behavior vary with the choice of bifurcation parameter values, and the exact space of inappropriate initial conditions is yet to be strictly defined.

3.4 Case 4 (phase lock / broadband w/out fundamental frequency): \( \beta = 3.7, \gamma = 0.411 \)

In this case we are testing at significantly high \( \beta \) values (3.7) and \( \gamma \) values (0.411) and expect to see some of the complicated behaviors associated with chaos. In fact, the behaviors are sufficiently complex as to make basin identification more difficult than in the previous cases. In Figs. 5(a-d) we demonstrate this difficulty with the nonuniformity of the basins as delineated by four different solution characterizations: \( u'v' \) correlation, kinetic energy, skewness, and second-order structure function. In the following paragraphs we will discuss the observed discrepancies among these parameters and determine which, if any, are correct.

Here we have implemented testing of random initial conditions along the \( a^0 = \beta^p \) and \( a^0 = 1 - \beta^p \) axes. We have found that for the many I.C.s we have tested power spectra always indicate the same type of behavior, that of broadband without the fundamental frequency (bottom of Fig. 5(e)). We also show an example of a time series associated with this basin at the bottom of Fig. 5(f). In addition to this basin, divergent trajectories, as in the previous case, are identified in the quadrilateral basin in the upper right-hand corner of the I.C. grid. As we previously mentioned the divergent region is assigned a \( u'v' \) correlation of positive one, and surprisingly the chaotic basin along the diagonals displays complete correlation of the \( u \)- and \( v \)-component velocities, leading to the failure of the \( u'v' \) correlation to distinguish between these basins. The reason for the perfect correlation within the chaotic region will be illuminated shortly.

The basin mapping using the kinetic energy contours shown in Fig. 5(b), seems to be more reasonable in that it displays three types of behavior. In addition, Figs. 5(c) and (d) display four individual basins and are qualitatively indistinguishable from one another. To evaluate the accuracy of these basins we first look at the value of the kinetic energy (Fig.
along the $a^0 = b^0$ and $a^0 = 1 - b^0$ axes. We see that this value varies somewhat, indicating that the kinetic energy is SIC for chaotic trajectories, as we would expect. As can be seen from Fig. 5(b) this basin is still distinguishable in this case. Otherwise the differences between Fig. 5(b) and Figs. 5(c,d) amount to the kinetic energy distinguishing only one other type of behavior (excluding divergence) while the skewness and second-order structure function of the $u$-component velocity distinguish two additional types.

![Figure 5: Contours for $\beta = 3.7, \gamma = 0.411$ using (a) $u'v'$ correlation, (b) kinetic energy, (c) skewness of $u$-component velocity, (d) second-order structure function of $u$-component velocity along with (e) power spectra and f) example time series for each respective attractor.](image)

The identical power spectra for the two intertwined basins making up the main portion of Figs. 5(c) and (d) are shown in the top two plots of Fig. 5(e) and indicate phase-locked trajectories. Also, the $u$-component velocity time series for one trajectory chosen from within each of these basins are shown in the top two plots of Fig. 5(f). These basins have been determined by numerous tests to be the same solution with the $u$- and $v$-component velocities switching values between basins. That is, one of the velocities always takes on the oscillations of the top time series, and one always oscillates as the middle time series; distinguishable basins indicate a reversal of this preference. In the case of kinetic energy it is found that differences between numerical values in the two phase-locked basins are on the order of $10^{-5}$ (due to round-off error and initial transients)—two orders of magnitude less than its variation along the chaotic attractor and thus below the plotting resolution. On the
other hand, the variations of the numerical values of skewness and second-order structure function (both based on only one component of the velocity) between the two phase-locked basins are generally two orders of magnitude greater than the variations of these solution parameters’ numerical values along the chaotic attractor and are thus visible. This method of interpreting the basin plots and testing points within each basin for consistency was alluded to previously. Further discussion in the following sections will provide an explanation of the proper parameters to be used in future studies.

3.5 Case 5 (noisy phase lock / noisy quasiperiodic w/ fundamental): $\beta = 3.6355, \gamma = 0.45$

Case 4 is sufficient evidence that further studies of the consistency of solution parameters, the circumstances for which a specified parameter will be correct, and how to avoid misidentified solution basins are needed. In case five we consider only the kinetic energy and flatness contours, displayed in Figs. 6(a) and (b), respectively.

![Figure 6](image_url)

Figure 6: Contours for $\beta = 3.6355, \gamma = 0.45$ using (a) kinetic energy and (b) flatness along with the (c) power spectra and (d), (e) example component time series.

Figure 6(a) is reminiscent of Fig. 5(b) in that the value of the kinetic energy varies along the diagonals and shows little change in the remainder of the nondivergent basin. Also, Fig. 6(b) is similar to Figs. 6(c,d) in that it identifies an additional solution basin along with those identified by the kinetic energy. These observations lead us to expect that some phase difference might be responsible for the inconsistent basin mappings.
Case 5 displays characteristics common to most basin plots identified in this study: “lines” of solutions along the diagonals and asymmetry, in this case, (but most often symmetry) about the $a^0 = b^0$ and, sometimes, the $a^1 = 1 - b^0$ axes. In this case these axes (with some exceptions along $a^0 = 1 - b^0$) are determined to correspond to noisy quasiperiodicity with the fundamental frequency still present (bottom plots of Figs. 6(c–e)). The present asymmetry of the flatness contours, Fig. 6(b), is seen for basins of noisy phase-locked behavior, distinguished by the power spectra and time series of the top and middle plots of Fig. 6(c–e), and is again due to the reversal of behavior of individual velocity components. These generalities are worthy of our attention, and here we focus on the causation of such peculiarities.

Consider again the 2-D PMNS equation rewritten here as:

$$a^{(n+1)} = a^{(n)} \beta_u \left( 1 - a^{(n)} \right) - \gamma_u a^{(n)} b^{(n)}, \quad (14a)$$

$$b^{(n+1)} = b^{(n)} \beta_v \left( 1 - b^{(n)} \right) - \gamma_v a^{(n)} b^{(n)}. \quad (14b)$$

To study the solution along the primary diagonal we set $a^0 = b^0$. From Eqs. (12) we see that when we restrict the bifurcation parameters such that $\beta_u = \beta_v$ and $\gamma_u = \gamma_v$ we are left with identical equations now of the form:

$$u^{(n+1)} = \beta u^{(n)} \left( 1 - u^{(n)} \right) - \gamma u^{(n)} u^{(n)}. \quad (15)$$

This is responsible for the unique lines of solutions along $a^0 = b^0$ and also explains why the $uv$ correlation is always positive along this line, independent of solution type. The same type of analysis can be performed for the line of I.C.s along $a^0 = 1 - b^0$, and one finds identical equations of the form:

$$u^{(n+1)} = u^{(n)} \beta \left( 1 - u^{(n)} \right) - \gamma u^{(n)} \left( 1 - u^{(n)} \right), \quad (16a)$$

$$= \left( \beta - \gamma \right) u^{(n)} \left( 1 - u^{(n)} \right). \quad (16b)$$

To investigate in a more general manner the occurrence of symmetries in the basin mappings of this study we begin with the general formula for the coordinates of I.C.s symmetric about the $a^0 = b^0$ axis:

$$a_1^0 = b_2^0, \quad (17a)$$

$$b_1^0 = a_2^0. \quad (17b)$$

Here subscripts are the initial condition number and superscripts are the usual iteration counter. By substitution of Eqs. (17) into Eqs. (14), along with the bifurcation parameter restrictions, $\beta_u = \beta_v$ and $\gamma_u = \gamma_v$, we derive the DDS for each initial condition:

$$a_1^{(n+1)} = a_1^{(n)} \beta \left( 1 - a_1^{(n)} \right) - \gamma a_1^{(n)} b_1^{(n)}, \quad (18a)$$

$$b_1^{(n+1)} = b_1^{(n)} \beta \left( 1 - b_1^{(n)} \right) - \gamma a_1^{(n)} b_1^{(n)}. \quad (18b)$$
\[ a_2^{(n+1)} = b_1^{(n)} \beta \left(1 - b_1^{(n)}\right) - \gamma a_1^{(n)} b_1^{(n)}, \quad (19a) \]

\[ b_2^{(n+1)} = a_1^{(n)} \beta \left(1 - a_1^{(n)}\right) - \gamma a_1^{(n)} b_1^{(n)}. \quad (19b) \]

From Eqs. (18) and (19) we see that if the initial conditions are given by Eqs. (17) then

\[ a_1^{(n+1)} = a_2^{(n+1)} \]  

\[ a_2^{(n+1)} = b_1^{(n+1)}, \]

and these are independent of \( \beta \) and \( \gamma \). This indicates that symmetry is necessary for all coupled bifurcation parameter values and, neglecting variations due to round-off errors, periodic trajectories are necessarily 180 degrees out-of-phase.

It is now advisable to reconsider case 4 with this new insight. The obvious question is why do symmetric I.C.s not display symmetric mappings using the skewness and second-order structure function. The answer to this question is found by looking at the entire history of the variables. For a short period, sometimes under 100 iterations, the velocity components act as predicted by Eqs. (18) and (19). Machine round-off error is then responsible for small deviations from this symmetry. The new transients often lead back to oppositely phased attractors, corresponding to an attracting fixed point of Eqs. (3). This phase difference, as noted before, accounts for the discrepancies in the values of the skewness and second-order structure function (along with the other velocity component specific parameters) and is responsible for the inconsistent mappings of Figs. 5(b–d). Therefore it is advisable that either the kinetic energy contours, or component independent formulations of other solution characterization parameters, be used in future studies and simulations as the characterizations should be machine/transient independent.

3.6 Case 6 (noisy quasiperiodic w/o fundamental / broadband w fundamental): \( \beta = 3.75, \gamma = 0.099 \)

By inspection of Figs. 1(a–d) it is hard to distinguish what types of behavior, exactly, will be seen for these values of \( \beta \) and \( \gamma \), but we can expect that the behavior will be quite complicated. Indeed the trajectories for this case are ultimately determined to be noisy quasiperiodic without the fundamental frequency and broadband with the fundamental frequency.

Considering the results of the previous cases, we choose to employ the kinetic energy contours as the distinguishing characterization in the present case. The three basins shown in Fig. 7(a) correspond to noisy quasiperiodic without the fundamental frequency present \( (a^0 = \theta^0 \text{ and } b^0 = 1 - \theta^0 \text{ axes}) \), broadband with the fundamental frequency present (most of unit square), and divergent (upper right-hand corner) trajectories. We observe that the power spectrum in the upper plot of Fig. 7(b) might easily be called broadband (w/o fundamental), but more careful inspection indicates remnants of spectral peaks at normalized frequencies near 0.4 and 0.8, thus implying quasiperiodicity. From numerous tests of I.C.s within this region we have established that Fig. 7(a) is an accurate basin mapping. Though the kinetic energy contours do exhibit a sensitivity to initial conditions in the chaotic basins, that sensitivity is small compared to the differences in the average values between the noisy quasiperiodic basin and the broadband with the fundamental frequency basin. An example of the \( u \)-component velocity time series for both the noisy quasiperiodic and the broadband attractors are shown in Fig. 7(c). One should also notice that the area of divergence is considerably more narrow than in the previous two cases at higher values of \( \gamma \) and lower
values of $\beta$. This suggests that divergence depends more heavily on the value of $\gamma$ than $\beta$, in agreement with Figs. 1(a–d).

3.7 Case 7 (Broadband w/ fundamental / Broadband w/o fundamental): $\beta_{u,v} = 3.83$, $\gamma_{u,v} = -0.0678$

The first thing one notices when examining the basin plots from various solution characterization parameters for case 7 is that seemingly no two of the parameters act alike. Here we arbitrarily choose to show the basin identification obtained from the values of the second-order structure function in Fig. 7(a). One structure commonly identified by all parameters, again, is the basin corresponding to the $a^0 = b^0$ and $a^0 = 1 - b^0$ axes. This, then, is the first basin we investigate in depth. Indeed, all points tested within this basin are of the distinct type shown by the top power spectrum of Fig. 8(b) and the top $u$-component velocity time series of Fig. 8(c). This type of behavior is known as periodic with a different fundamental, and it produces velocity components that oscillate between three values (not readily detected on the scale of the plot, but easily seen in a phase portrait).

Next we investigate the set of I.C.s surrounding the periodic basin that the identification parameters are confusing. Of the many points we have tested within this region, we have seen two distinct types of power spectra. The first type is known as broadband with a different fundamental frequency and is shown in the middle of Fig. 8(b). The second type is seen on the bottom of Fig. 8(b) and is known as broadband without the fundamental frequency present.

From consideration of the time series for these behaviors (middle and bottom of Fig. 8(c)) we gain insight into why the solution characterization parameters misbehave. First, notice
that both of the chaotic trajectories can approximate very closely the periodic trajectories for extended periods of time (left half of plots in Fig. 8(c)) which are followed by “bursts” of turbulent behavior. This behavior has been associated with type III intermittency by McDonough and Huang [2001]. From this it should be quite obvious that long term statistics will be adversely affected by this type of behavior. In fact the contrasting power spectra can be produced by the same time series by varying the sample size or location for use in construction of the PSD plot. From looking at time series on the order of 10^6 iterations we have witnessed no relaminarization of the velocity field and continued turbulent “bursts” over a large sample of the tested initial conditions.

4. Conclusions

In this study we have performed a basic investigation of a novel subgrid-scale “synthetic velocity” model to be used in Additive Turbulent Decomposition turbulence simulations. It has been shown that identification of the types of solutions of this DDS, known as the ‘Poor Man’s Navier Stokes’ equation, is possible in an automatic calculation without overwhelming computational storage or time using various DDS, statistical, and turbulence properties as solution-type characterization parameters.

We have shown through bifurcation diagrams, power spectra, and component velocity time series that the 2-D PMNS equation can produce thirteen different types of behavior, along with divergence, for bifurcation values in the ranges of 1.0 ≤ β ≤ 4.0 and −1.1 ≤ γ ≤ 4.0. A wide array of bifurcation sequences, some confirmed by experiments
in turbulence and some as yet unseen, have been shown to be obtainable by variation of bifurcation parameter values and/or initial conditions indicating the expected flexibility of this DDS to model nonlocal variables and, by inference, the possibility of the N.-S. equation to produce nonunique solutions. This large range of behaviors, even in 2-D, indicates the potential for this DDS to be successful in replicating the small scale fluctuations of turbulent velocity components. In fact, the method of McDonough et al. [1998] has been successfully applied to 2-D LDV data by Yang et al. [2001] in the context of the 2-D PMNS equation.

The most important findings in terms of the implication to the curve fitting procedure have been the derivation of the necessary symmetry due to the coupled bifurcation parameters of this study and the variation of the regions of divergence as bifurcation parameters and/or initial conditions are varied. Future studies in the much more important case of the 3-D PMNS equation have been augmented by the determination of selective criteria to be used in solution characterization and in general a basic understanding of the tools necessary in investigating the effects of initial conditions on DDSs of this nature.

References


