Chapter 4 – The Wave Equation and its Solution in Gases and Liquids

Slides to accompany lectures in Vibro-Acoustic Design in Mechanical Systems
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Chapter 4 – The Wave Equation and its Solution in Gases and Liquids

Simplifying Assumptions
- The medium is homogenous and isotropic
- The medium is linearly elastic
- Viscous losses are negligible
- Heat transfer in the medium can be ignored
- Gravitational effects can be ignored
- Acoustic disturbances are small

Sound Pressure and Particle Velocity

\[ p(\vec{r},t) = p_0 + p(\vec{r},t) \]

Total Pressure  Undisturbed Pressure  Disturbed Pressure

Particle Velocity

\[ \vec{u}(\vec{r},t) = u_x \hat{i} + u_y \hat{j} + u_z \hat{k} \]

Density and Temperature

\[ \rho(\vec{r},t) = \rho_0 + \rho(\vec{r},t) \]

Total Density  Undisturbed Density  Disturbed Density

Absolute Temperature

\[ T(\vec{r},t) \]

Equation of Continuity

\[ \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \rho u_x \frac{\partial u_x}{\partial x} \Delta x \Delta y \Delta z - \left( \rho + \frac{\partial \rho}{\partial x} \Delta x \right) \left( u_x + \frac{\partial u_x}{\partial x} \right) \Delta y \Delta z \]
Equation of Continuity

\[ \frac{\partial}{\partial t} \left( \rho \Delta x \Delta y \Delta z \right) = \rho \frac{\partial u}{\partial x} \Delta x \Delta y \Delta z \]

\[ \frac{\partial}{\partial t} \left( \rho u \Delta x \Delta y \Delta z \right) = \rho u \frac{\partial \Delta x}{\partial x} \Delta x \Delta y \Delta z \]

3D Equation of Continuity

\[ \frac{\partial \rho}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = 0 \]

Equation of Motion

\[ F = (p + p)(\rho \Delta x \Delta y \Delta z) = \left( p + p + \frac{\partial (p + p)}{\partial x} \Delta x \right) \Delta y \Delta z \]

\[ = \frac{\partial (p + p)}{\partial x} \Delta x \Delta y \Delta z \]

\[ = -\frac{\partial p}{\partial x} \Delta x \Delta y \Delta z \]

Equation of Motion

\[ a_i = \lim_{L \to 0} \frac{u_i (x + \Delta x, t + \Delta t) - u_i (x, t)}{\Delta t} \]

\[ a_i = \lim_{L \to 0} \frac{u_i (x + \Delta x - x) + \frac{\partial u_i}{\partial t} (t + \Delta t - t) + \cdots}{\Delta t} \]

Assuming small disturbances the 1st term is small.

\[ a_i = \frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial t} \]
Equation of Motion

\[ a = \frac{\partial u}{\partial t} \]
\[ F_i = -\frac{\partial}{\partial x_i} \Delta u \Delta \gamma \Delta c \]
\[ \Delta u = \frac{(\rho_0 + \rho) \Delta u \Delta \gamma \Delta c}{\partial t} \]
\[ \rho \frac{\partial \Delta u}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \]

3D Equation of Motion

\[ \rho \frac{\partial \Delta u}{\partial t} + \nabla \rho = 0 \]

Thermodynamic Equation of State

For an adiabatic process:

\[ \gamma \rho \frac{\partial u}{\partial t} - \rho \frac{\partial p}{\partial x} = 0 \]

For a fluid of fixed mass:

\[ M = \rho V_0 \]
\[ M = (\rho_0 + \rho)(V_0 + \delta V) \]
\[ = \rho_0 V_0 + V_0 \rho + \rho \delta V \]
\[ V_0 \rho + \rho \delta V = 0 \]
\[ \rho = -\rho_0 \frac{\delta V}{V_0} \]

Equation of Motion

For an Ideal Gas

Equation of State:

\[ \frac{p}{\rho} = RT \]

For an adiabatic process:

\[ \frac{p_0 + p}{p_0} = \left( \frac{\rho_0 + \rho}{\rho_0} \right)^{\gamma} \]
\[ \gamma = \frac{c_s}{c} \]
\[ \frac{\partial p}{\partial \rho} \mid_{\gamma} = \rho \left( \frac{\rho_0 + \rho}{\rho_0} \right)^{\gamma - 1} = \frac{\rho_0}{p_0} = \gamma RT_0 \]

Thermodynamic Equation of State

The pressure-density relation for a fluid is usually non-linear:

\[ p = \rho \rho_0 \rho_0 \]
\[ p = \rho \rho_0 \rho_0 \]
\[ p = \rho_0 \rho_0 \]
\[ \rho = \rho_0 \rho_0 \]
\[ \rho_0 \rho_0 \]
\[ \rho \rho_0 \rho_0 \]

For small (acoustic) changes about the ambient state:

\[ p = \rho \rho_0 \rho_0 \]
\[ \rho = \rho_0 \rho_0 \]
\[ \rho_0 \rho_0 \]
\[ \rho \rho_0 \rho_0 \]
\[ \rho_0 \rho_0 \]
\[ \rho \rho_0 \rho_0 \]

The Speed of Sound

Units of \( \frac{p}{\rho} \) kg/m/s \( \frac{N}{m^2} \) \( \frac{kg}{m^3} \) m/s \( \frac{m}{s} \)

\[ c = \sqrt{\frac{p}{\rho}} \]

For an ideal gas undergoing adiabatic compression and expansion:

\[ c = \frac{\sqrt{\frac{p}{\rho}}}{\sqrt{\frac{p_0}{\rho_0}}} = \frac{\rho_0}{\rho} \frac{p}{p_0} \frac{RT_0}{RT} \]

Mass of a mole of gas

\[ R = 8.315 \frac{J}{mol K} \]
The Speed of Sound

\[ c = \sqrt{\frac{RT}{M}} \]

For:  
\[ \rho_0 = 1.013 \times 10^5 \text{ Pa} \] (one atmosphere)  
\[ \rho_0 = 1.21 \text{ kg/m}^3 \] (air at 20°C)  
\[ \gamma = 1.402 \text{ (air)} \]

\[ c = \sqrt{\frac{1.402 \times 10^3 \text{ m}^3}{1.21 \text{ s}}} = 343.0 \text{ m/s} \]

Deriving the Wave Equation

The Continuity Equation

\[ \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0 \]

Differentiate with time

\[ \frac{\partial^2 \rho}{\partial t^2} + \rho_0 \frac{\partial^2 u}{\partial x \partial t} = 0 \]

The Equation of Motion

\[ \rho_0 \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial p}{\partial x} = 0 \]

\[ \rho_0 \frac{\partial^2 u}{\partial x^2} + \frac{\partial p}{\partial x} = 0 \]

Subtract first from second equation

\[ \rho_0 \frac{\partial^2 u}{\partial x} - \frac{\partial^2 p}{\partial x^2} = 0 \]

Use equation of state to eliminate \( \rho \)

\[ \beta = \rho_0 \frac{\partial p}{\partial \rho_0} \]

or

\[ \rho = \beta \frac{\partial p}{\partial \rho_0} \]

Then

\[ \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0 \]
For the wave at \( \rho \)

\[ \frac{\partial^2 p}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial x^2} = 0 \]

In 3D

\[ \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \]

**The Wave Equation**

In 1D

\[ \frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \]

In 3D

\[ \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \]

**Solutions to the Wave Equation**

In One Dimension

Piston oscillates with frequency \( f \)

\[ p(x, t) = P \sin(\omega t - kx) \]

Higher pressure (compression)

\[ p(x, t) = P \sin(\omega t + kx) \]

Lower pressure (rarefaction)

\[ p(x, t) = P \sin(\omega t - kx) \]

Waves move with speed of sound \( c \)

**Functional Description**

For the wave at \( t = 0 \): \( p(x) = P \sin(\frac{2\pi}{\lambda} x) \)

Wavelength \( \lambda = \frac{c}{f} \)

For the wave at \( t = \lambda / c \): \( p(x, \lambda / c) = P \sin\left(\frac{2\pi}{\lambda} (x - \lambda)\right) \)

Wavenumber \( \kappa = \frac{\omega}{c} \)

For the wave at any \( t \): \( p(x, t) = P \sin\left(\frac{2\pi}{\lambda} (x - \lambda t)\right) \)

**Acoustic Particle Velocity**

**Equation of Motion**

\[ \rho \ddot{u} = -\frac{dp(x, t)}{dx} \]  

NO FLOW!! Fluid particles only oscillate! (series of springs and masses)

\[ u(x, t) = \frac{1}{\rho} \int_{x'}^x dp(x', t) dx' = \frac{P_k}{\rho} \cos(kx - \omega t) \]

\[ u(x, t) = \frac{P_k}{\rho} \sin(kx - \omega t) \]

\[ u(x, t) = \frac{P_k}{\rho} \sin(kx - \omega t) \]  

(i.e., a momentum wave in phase with the sound pressure wave)

**Exponential Form**

Recall that any harmonic function may be expressed as either the real or imaginary part of a similar but complex function, e.g.,

\[ F \cos(\omega t) = \text{Re}[F e^{j\omega t}] \quad \text{or} \quad F \sin(\omega t) = \text{Im}[F e^{j\omega t}] \]

\[ e^{j\theta} = \cos\theta + j \sin\theta \]

\[ e^{j(\omega + \theta)} = \cos(\omega t + \theta) + j \sin(\omega t + \theta) \]

\[ e^{j(\omega + \theta)} = e^{j\theta} e^{j\omega t} \]

(This is a plane harmonic wave traveling in the +\( \theta \) direction.)

\[ e^{j(\omega + \theta)} = e^{j\omega t} e^{j\theta} \]

(This is a plane harmonic wave traveling in the +\( \theta \) direction.)

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Harmonic Solution of 1D Wave Equation

For harmonic waves:

\[ p(x,t) = \rho \left( \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} \right) \]

Substituting:

\[ u(x,t) = e^{j \omega t} \]

\[ \frac{d^2 p(x,t)}{dx^2} + k^2 p(x,t) = 0 \]

Solution:

\[ p(x) = C e^{j k x} + D e^{-j k x} \]

Particles Velocity:

\[ \ddot{u}(x,t) = \frac{\rho C}{\rho_0 c} e^{j (\omega x - ct)} - \frac{\rho C}{\rho_0 c} e^{-j (\omega x + ct)} \]

Example:

\[ u_0 = \cos(\omega t) \]

\[ u_L = 0 \]

\[ \ddot{u}(x,t) = \frac{p_r}{\rho_0 c} e^{j (\omega x - ct)} - \frac{p_r}{\rho_0 c} e^{-j (\omega x + ct)} \]

\[ 1 = \frac{p_r}{\rho_0 c} - \frac{p_r}{\rho_0 c} e^{2 j \omega t} \]

\[ p_r = \frac{\rho_0 c}{1 - e^{2 j \omega t}} \]

\[ p_r = \frac{p_r}{\rho_0 c} e^{j \omega L} - \frac{p_r}{\rho_0 c} e^{-j \omega L} \]

Specific Impedance:

\[ Z = \frac{p}{u} \]

For the free plane wave case (no reflected wave):

\[ Z_0 = \rho_0 c \]
Sound Power Level:

Recall:

Example:

\[ P'(x,t) = p_0 e^{j(\omega t + kx)} \]

\[ \vec{u}(x,t) = \frac{p_0}{\rho c} e^{j(\omega t - kx)} \]

Reference Quantities and dB Scale

Recall:

\[ L_p(dB) = 10 \log \left( \frac{P}{P_0} \right) \]

\( P_0 = 20 \mu Pa \)

Sound Power Level:

\[ L_p(dB) = 10 \log \left( \frac{W}{W_0} \right) \]

\( W_0 = 1 \times 10^{-12} \text{ watts} \)

Example: What are the sound pressure and sound levels of the fan in the previous problem?

\[ L_p = 20 \log \left( \frac{3.72}{20 \times 10^{-5}} \right) = 102.4 \text{ dB (re } 20 \mu Pa) \]

\[ L_p = 10 \log \left( \frac{0.001}{10^{-10}} \right) = 90 \text{ dB (re } 10^{-12} \text{ W}) \]

Approximate Relationship between \( L_p \) and \( L_m \)

\[ \frac{P_m}{W} = \frac{P_m}{P_0} \]

\[ L_p = 10 \log \left( \frac{p_0^2}{W} \right) \]

\[ p_0^2 / W = 10 \log \left( \frac{W}{W_0} \right) \]

\[ p_m / p_0 = (20 \times 10^{-5} \text{ Pa}) \]

\[ 4.15 \times 10^{-14} \text{ watts} = W_{m} \]

\[ L_p = 10 \log \left( \frac{W_{m}}{W_0} \right) = 10 \log \left( \frac{W}{W_0} \right) \]

\[ L_m = L_p - 10 \log (S) \text{ (in } m^2) \]

Comparing to the previous (exact) calculation:

\[ L_m = 90 - 10 \log (0.06) = 102.2 \text{ dB (re } 20 \mu Pa) \]
Harmonic Solution for Free Spherical Wave

Sound pressure a distance \( r \) from the point source:

\[
\rho(r,t) = \frac{A_i}{r} e^{i(kr-\omega t)}
\]

Free field (no reflections)

This is similar to a plane wave, but for spherical waves the sound pressure amplitude decreases with distance from the source of sound.

Impedance of a Free Spherical Wave

\[
Z = \frac{p}{u_i} = \frac{\rho_i c}{1 + \frac{i}{kr}}
\]

For \( kr \ll 1 \) (the nearfield)
\[
Z(r) = j\rho_i c k r
\]

For \( kr \gg 1 \) (the farfield)
\[
Z(r) = \rho_i c
\]

Sound Intensity and Sound Power

\[
I_r(r) = \frac{1}{2} \text{Re}(pu) = \frac{A_i^2}{2\rho_i c r^2}
\]
\[
W = T(r) 4\pi r^2 = 2\pi \frac{A_i^2}{\rho_i c}
\]

Imaginary sphere enclosing source