AC Steady-State Analysis

Sinusoidal Forcing Functions,
Phasors, and Impedance

The Sinusoidal Function

Terms for describing sinusoids:

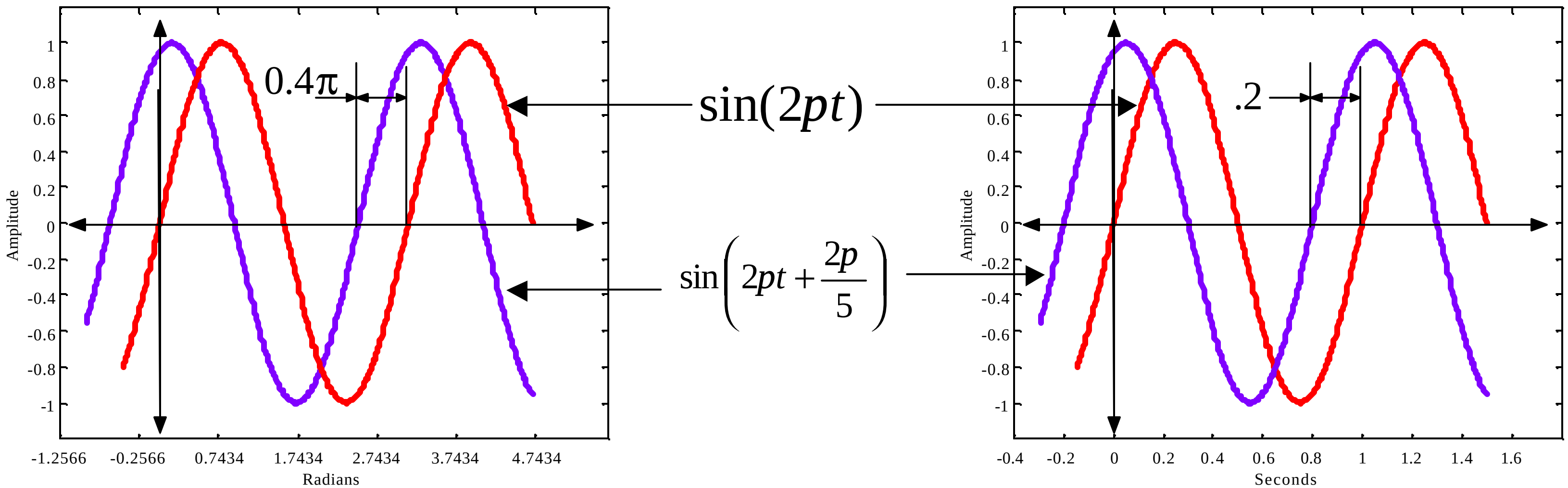
$$x(t) = X_m \sin(wt + q) = X_m \sin(2pft + q)$$

Maximum Value, Amplitude, or Magnitude

Radian Frequency in Radian/second

Phase

Frequency in cycles/second or Hertz (Hz)



Trigonometric Identities

$$\cos(wt) = \sin\left(wt + \frac{p}{2}\right) \qquad \sin(wt) = \cos\left(wt - \frac{p}{2}\right)$$

$$-\cos(wt) = \cos\left(wt \pm p\ (\mathrm{or}\ 180^\circ)\right) \qquad -\sin(wt) = \sin\left(wt \pm p\ (\mathrm{or}\ 180^\circ)\right)$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

Radian to degree conversion

multiply by 180/p

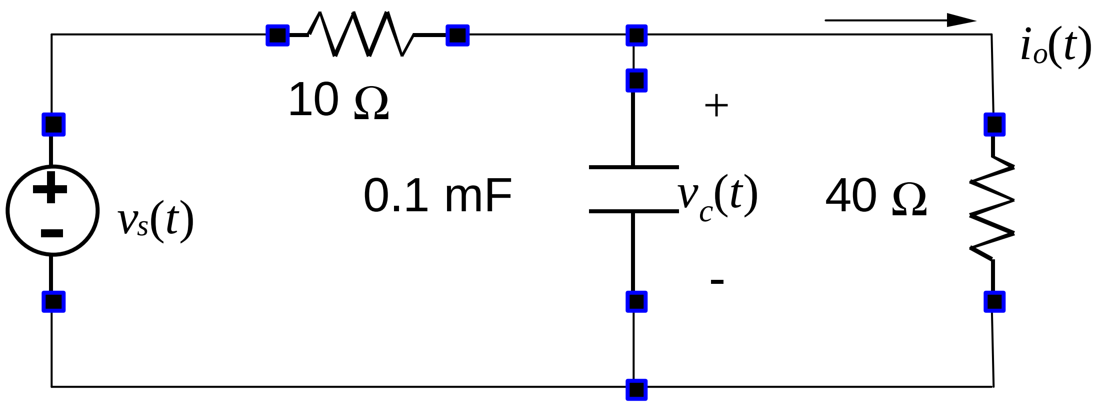
Degree to radian conversion
multiply by p**/180**

$$X_m \sin(w\,t \pm q) = X_m \cos(q)\sin(w\,t) \pm X_m \sin(q)\cos(w\,t)$$

$$A\cos(w\,t) + B\sin(w\,t) = \sqrt{A^2 + B^2}\cos\left(w\,t + \tan^{-1}\left(\frac{-B}{A}\right)\right)$$

Sinusoidal Forcing Functions

Determine the forced response for $i_o(t)$ the circuit below with $v_s(t) = 50\cos(1250t)$:



Note:

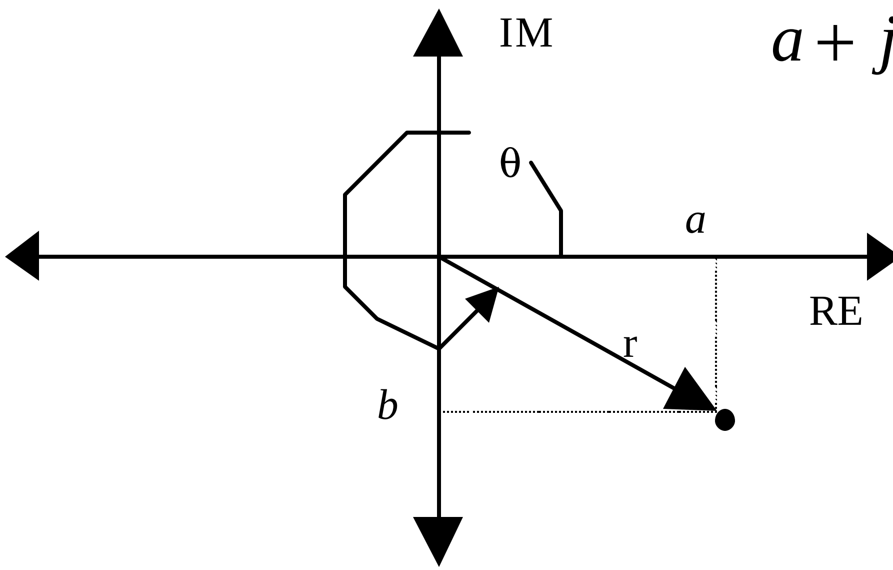
$$20\text{m}v_s = 0.8\text{m}\frac{di_o}{dt} + i_o$$

Show:

$$i_o(t) = \frac{1}{2}\left(\sin(1250t) + \cos(1250t)\right) = \frac{1}{\sqrt{2}}\cos\left(1250t - \frac{p}{4}\right)$$

Complex Numbers

Each point in the complex number plane can be represented by in a Cartesian or polar format.



$$a + jb = r\exp(jq) = r\angle q$$

$$r = \sqrt{a^2 + b^2}$$

$$q = \tan^{-1}\left(\frac{b}{a}\right)$$

$$a = r\cos(q)$$

$$b = r\sin(q)$$

Complex Arithmetic

Addition:

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

Multiplication and Division:

$$(r\angle q)(v\angle f) = rv\angle(q + f) \qquad \frac{r\angle q}{v\angle f} = \frac{r}{v}\angle(q - f)$$

Simple conversions:

$$j = 1\angle 90^{\circ}, \qquad \frac{1}{j} = -j, \qquad \text{-}1 = \angle 180^{\circ}$$

Euler's Formula

Show: $$\exp(jq) = \cos(q) + j\sin(q)$$

A series expansion ….

$$\exp(jq) = 1 + \frac{jq}{1!} - \frac{q^2}{2!} - \frac{jq^3}{3!} + \frac{q^4}{4!} + \frac{jq^5}{5!} - \cdots$$

$$\cos(q) = 1 - \frac{q^2}{2!} + \frac{q^4}{4!} - \frac{q^6}{6!} + \cdots$$

$$\sin(q) = \frac{q}{1!} - \frac{q^3}{3!} + \frac{q^5}{5!} - \frac{q^7}{7!} + \cdots$$

Complex Forcing Function

Consider a sinusoidal forcing function given as a complex function:

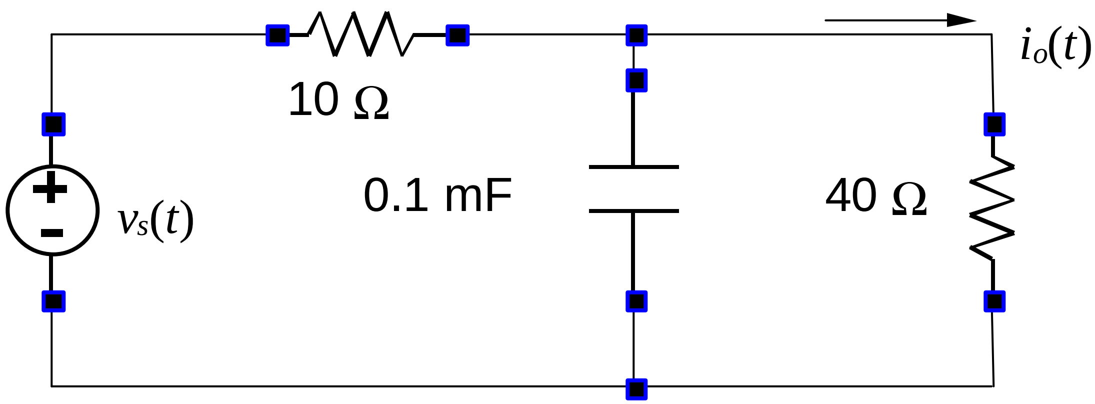
$$X_m \exp(j(wt+q)) = X_m \cos(wt+q) + jX_m \sin(wt+q)$$

- Based on a signal and system's concept (orthogonality), it can be shown that for a linear system, the ***real part of the forcing function only affects the real part of the response*** and the ***imaginary part of the forcing function only affect the imaginary part of the response***.
- For a linear system excited by a sinusoidal function, the steady-state response everywhere in the circuit will have the same frequency. Only the magnitude and phase of the response will vary.
- A useful factorization:

$$X_m \exp(j(wt+q)) = X_m \exp(jq)\exp(jwt)$$

Complex Forcing Function Example

Determine the forced response for $i_o(t)$ the circuit below with $v_s(t) = 50\exp(j1250t)$:



Note:

$$20\mathrm{m}v_S = 0.8\mathrm{m}\frac{di_o}{dt} + i_o$$

Show:

$$i_o(t) = \frac{1}{\sqrt{2}}\exp\left(j\left(1250t - \frac{p}{4}\right)\right)$$

$$\mathrm{Re}[i_o(t)] = \frac{1}{\sqrt{2}}\cos\left(1250t - \frac{p}{4}\right) \qquad \mathrm{Im}[i_o(t)] = \frac{1}{\sqrt{2}}\sin\left(1250t - \frac{p}{4}\right)$$

Phasors

Notation for sinusoidal functions in a circuit can be more efficient if the exp(-*jwt*) is dropped and just the magnitude and phase maintained via phasor notation:

Time Domain

$$x(t) = A\cos(wt + q)$$

$$x(t) = A\sin(wt + q)$$

Frequency Domain

$$\hat{X} = A\angle q$$

$$\hat{X} = A\angle\left(q - 90^\circ\right)$$

Impedance

The *w* affects the *resistive* force that inductors and capacitors have on the currents and voltages in the circuit. This generalized resistance, which affects both amplitude and phase of the sinusoid, will be called impedance. Impedance is a complex function of ω.

Given: $\hat{V} = A_V \exp(j(wt + q_V))$

$$\hat{I} = A_I \exp(j(wt + q_I))$$

using passive sign convention show:

For inductor relation $\hat{V} = \hat{Z}\hat{I}$:

Show $\hat{Z} = jwL$

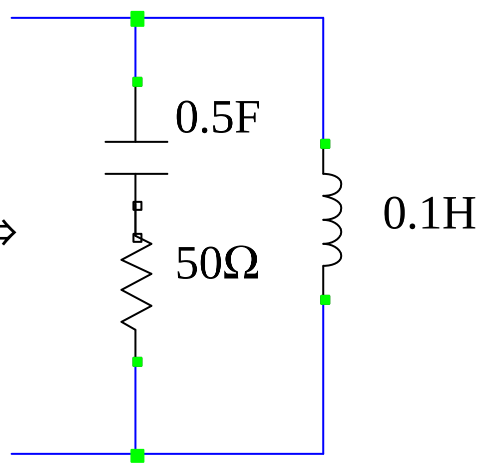
For capacitor relation $\hat{V} = \hat{Z}\hat{I}$:

Show $\hat{Z} = \dfrac{1}{jwC}$

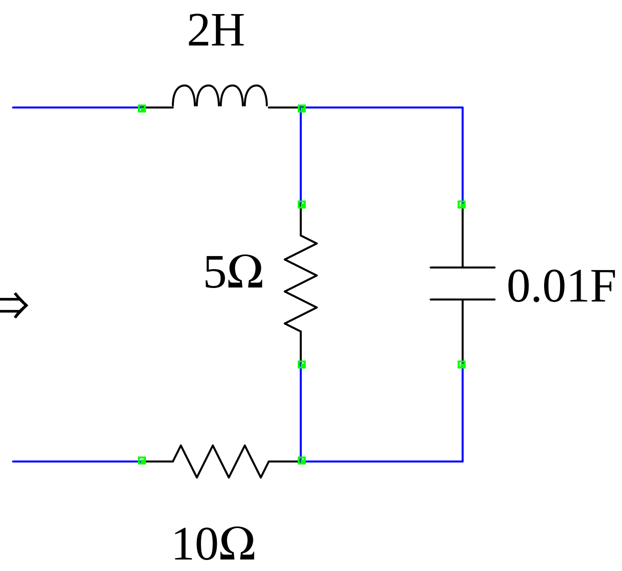
Finding Equivalent Impedance

Given a circuit to be analyzed for AC steady-state behavior, all inductors and capacitors can be converted to impedances and combined together as if they were resistors.

Find $Z_{eq} \Rightarrow$



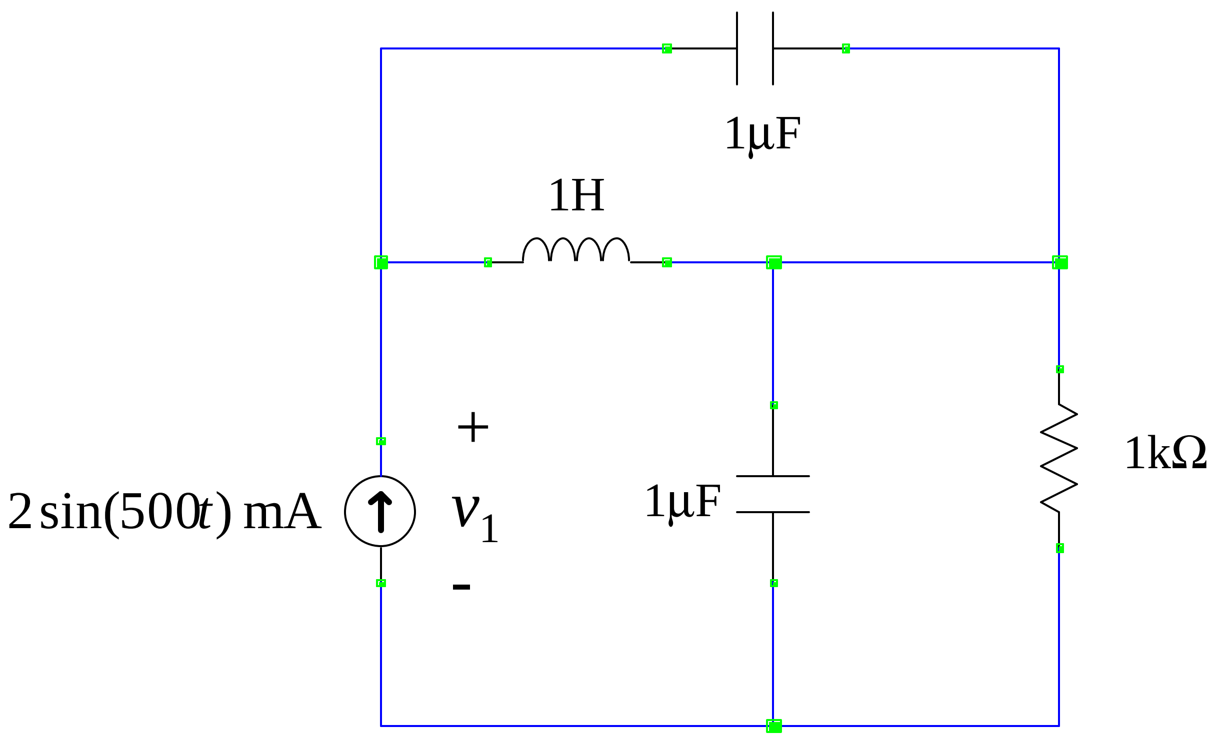
Find $Z_{eq} \Rightarrow$



Show $Z_{eq} = .2039\angle 79.6°$

Impedance Circuit Example

Find the AC steady-state value for $v_1(t)$:



Impedance Circuit Example

Find AC steady-state response for $i_o(t)$: