Members of a structural system are typically oriented in differing directions, e.g., Fig. 17.1. In order to perform an analysis, the element stiffness equations need to be expressed in a common coordinate system – typically the global coordinate system. Once the element equations are expressed in a common coordinate system, the equations for each element comprising the structure can be assembled.

Coordinate Transformations: Frame Elements
Consider frame element m of Fig. 17.7.

(a) Frame
(b) Local Coordinate End Forces and Displacements
Figure 17.7 shows that the local –
global coordinate transformations
can be expressed as
\[ x = \cos \theta X + \sin \theta Y \]
\[ y = -\sin \theta X + \cos \theta Y \]
and since \( z \) and \( Z \) are parallel, this
coordinate transformation is
expressed as
\[ z = Z \]
Using the above coordinate
transformations, the end force and
displacement transformations can
be expressed as
\[ \{Q\} = \{T\} \{F\} \] (17.11)

where \( \{Q\}_b = [Q_1 Q_2 Q_3]^T \) =
beginning node local coordinate
force vector; \( \{Q\}_e = [Q_4 Q_5 Q_6]^T \) =
end node local coordinate force
vector; \( \{F\}_b = [F_1 F_2 F_3]^T \) =
beginning node global coordinate
force vector; \( \{F\}_e = [F_4 F_5 F_6]^T \) =
end node global coordinate force
vector; \( [t] = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \)
local to global coordinate transfor-
mation matrix which is the same at
each end node for a straight
member; \( \{Q\} = [Q_b Q_e]^T \) =
element local coordinate force
vector; \( \{F\} = [F_b F_e]^T \) =
element global coordinate force
vector; and \( [T] = \begin{bmatrix} [t] & [0] \\ [0] & [t] \end{bmatrix} \) =
element local to global coordinate
transformation matrix.
Utilizing (17.9) for all six member
force components and expressing
the resulting transformations in
matrix form gives
\[ Q_X = \cos \theta F_X + \sin \theta F_Y \]
\[ Q_Y = -\sin \theta F_X + \cos \theta F_Y \] (17.9)
\[ Q_Z = F_Z \]
where \( x, X = 1 \) or \( 4 \); \( y, Y = 2 \) or \( 5 \);
and \( z, Z = 3 \) or \( 6 \).
The direction cosines used in the transformation matrices can easily be calculated from the nodal geometry, i.e.

\[
\cos \theta = \frac{X_e - X_b}{L} \\
\sin \theta = \frac{Y_e - Y_b}{L} \\
L = \sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}
\]

(17.13)

Since the end displacements are aligned with the end forces, the local to global coordinate displacement relationships are

\[
\{u\} = [T]\{v\}
\]

(17.14)

It is also useful in matrix structural analysis to calculate the global end displacements and forces in terms of the local coordinate end displacements and forces as shown in Fig. D.

Figure D: Global – Local Coordinate Relationships

Applying the global – local coordinate transformations to the end node forces gives

\[
\begin{align*}
F_X &= \cos \theta Q_X - \sin \theta Q_Y \\
F_Y &= \sin \theta Q_X + \cos \theta Q_Y \\
F_Z &= Q_Z
\end{align*}
\]

(17.15)

where \(X, x = 1 \text{ or } 4; Y, y = 2 \text{ or } 5; \) and \(Z, z = 3 \text{ or } 6.\)
or
\[
\{F\} = [T]^T \{Q\}
\] (17.17)

Similarly, the global coordinate displacement vector is related to the local coordinate displacement vector as
\[
\{v\} = [T]^T \{u\}
\] (17.18)

**Continuous Beam Members**

When analyzing continuous beam structures, the axial displacement and force degrees of freedom are typically ignored since they are zero unless there is axial loading and the continuous beam is restrained against longitudinal motion, i.e., is not free to expand. Regardless, in continuous beam structures the local and global coordinate systems typically coincide resulting in \([t_b] = [I]_{2\times2}\). This leads to
\[
\{F_b\} = \{Q_b\} \text{ and } \{v_b\} = \{u_b\}
\] (17.19)

**Truss Members**

Local – global force and displacement relationships (see Fig. 17.9):
\[
\begin{bmatrix}
Q_1 \\
Q_2
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4
\end{bmatrix}
\]

\[
\{Q_a\} = [T_a]^T \{F_a\}
\]

\[
\{u_a\} = [T_a]^T \{v_a\}
\]

Similarly,
\[
\{F_a\} = [T_a]^T \{Q_a\}
\]

\[
\{v_a\} = [T_a]^T \{u_a\}
\]

**Figure 17.9 Truss Member**

(a) Local Coordinate End Forces and Displacements for a Truss Member

(b) Global Coordinate End Forces and Displacements for a Truss Member
**MEMBER STIFFNESS RELATIONS IN GLOBAL COORDINATES**

To establish the global coordinate representation of the element stiffness equations, start by substituting \( \{u\} = [T] \{v\} \) into (17.4):

\[
\{Q\} = [k][T]\{v\} + \{Q_F\} \quad (17.a)
\]

which results in the force quantities being defined in the local coordinate system and the displacement vector expressed in terms of the global coordinate system. To transform the force vectors into the global coordinate system, pre-multiply both sides of (17.a) by \([T]^T\):

\[
[T]^T \{Q\} = [T]^T[k][T]\{v\} + [T]^T \{Q_F\} \quad (17.23)
\]

Substituting \( \{F\} = [T]^T \{Q\} \) into (17.23):

\[
\{F\} = [T]^T[k][T]\{v\} + \{F_F\} \quad (17.24)
\]

where \( \{F_F\} = [T]^T \{Q_F\} \).

Equation (17.24) in matrix form:

\[
\{F\} = [K]\{v\} + \{F_F\} \quad (17.25)
\]

where \([K] = [T]^T[k][T] = \text{global coordinate element stiffness}\).

For a continuous beam member (also see discussion following equation 1 and 17.19):

\[
[T] = [T_b] = [I]_{4x4}
\]

\[
\{v\} = \{v_b\} = \{u_b\}
\]

\[
\{F\} = \{F_b\} = \{Q_b\}
\]

\[
\{F_F\} = \{F_{fb}\} = \{Q_{fb}\}
\]

\[
[K] = [K_{bb}] = [k_{bb}] \quad \text{(see 17.6)}
\]

matrix, i.e., the element stiffness matrix coefficients aligned with the global coordinate system and

\[
K_{ij} = F_i|_{v_j=1}
\]

with all other \( v_k = 0 \) and \( k \neq j \).

All global coordinate stiffness equations are expressed by (17.24) and (17.25). However, for beam and truss structures, the transformation matrix \([T]\), displacement vector \( \{v\} \), and force vectors \( \{F\} \) and \( \{F_F\} \) must be for these members.
For a truss member (also see discussion following equations 1 and 17.21):

\[
[T] = [T_a]
\]

\[
\{v\} = \{v_a\}
\]

\[
\{F\} = \{F_a\}
\]

\[
\{F_f\} = \{F_{fa}\}
\]

\[
[K] = [K_{aa}] = [T_a]^T[k_{aa}][T_a]
\]

\[
= \frac{EA}{L} \begin{bmatrix}
  c^2 & cs & -c^2 & -cs \\
  cs & s^2 & -cs & -s^2 \\
  -c^2 & -cs & c^2 & cs \\
  -cs & -s^2 & cs & s^2
\end{bmatrix}
\]

(17.29)

\[c \equiv \cos \theta\]
\[s \equiv \sin \theta\]

**Structure Stiffness Relations**

The structure stiffness equations can now be determined using the global coordinate member stiffness equations. Generation of the structure stiffness equations is based on the three basic relationships of structural analysis: (1) equilibrium, (2) constitutive relationships, and (3) compatibility. Specifically, the direct stiffness procedure involves expressing:

(1) Node point equilibrium of the element end forces meeting at the node with the externally applied nodal forces;

(2) Substituting the global coordinate constitutive (matrix stiffness) equations for the forces in terms of the stiffness coefficients times the element end displacements and fixed-end force contributions; and

(3) Compatibility of the element end displacements with the structure displacement degrees of freedom \(\{d\}\).
Equilibrium Equations

\[ \sum F_x = P_1 - F_4^{(1)} - F_1^{(2)} \]
\[ \Rightarrow P_1 = F_4^{(1)} + F_1^{(2)} \]  
(17.30a)

\[ \sum F_y = P_2 - F_5^{(1)} - F_2^{(2)} \]
\[ \Rightarrow P_2 = F_5^{(1)} + F_2^{(2)} \]  
(17.30b)

\[ \sum M_z = P_3 - F_6^{(1)} - F_3^{(2)} \]
\[ \Rightarrow P_3 = F_6^{(1)} + F_3^{(2)} \]  
(17.30c)

where superscripts (1), (2) designate element (member) 1, 2; respectively.

are substituted into (17.30) to give

\[ P_1 = K_{11}^{(1)} v_1^{(1)} + K_{42}^{(1)} v_2^{(1)} + K_{43}^{(1)} v_3^{(1)} \]
\[ + K_{44}^{(1)} v_4^{(1)} + K_{45}^{(1)} v_5^{(1)} + K_{46}^{(1)} v_6^{(1)} + F_4^{(1)} \]
\[ + K_{11}^{(2)} v_1^{(2)} + K_{12}^{(2)} v_2^{(2)} + K_{13}^{(2)} v_3^{(2)} \]
\[ + K_{14}^{(2)} v_4^{(2)} + K_{15}^{(2)} v_5^{(2)} + K_{16}^{(2)} v_6^{(2)} + F_1^{(2)} \]  

\[ P_2 = K_{51}^{(1)} v_1^{(1)} + K_{52}^{(1)} v_2^{(1)} + K_{53}^{(1)} v_3^{(1)} \]
\[ + K_{54}^{(1)} v_4^{(1)} + K_{55}^{(1)} v_5^{(1)} + K_{56}^{(1)} v_6^{(1)} + F_4^{(1)} \]
\[ + K_{21}^{(2)} v_1^{(2)} + K_{22}^{(2)} v_2^{(2)} + K_{23}^{(2)} v_3^{(2)} \]
\[ + K_{24}^{(2)} v_4^{(2)} + K_{25}^{(2)} v_5^{(2)} + K_{26}^{(2)} v_6^{(2)} + F_1^{(2)} \]  

\[ P_3 = K_{61}^{(1)} v_1^{(1)} + K_{62}^{(1)} v_2^{(1)} + K_{63}^{(1)} v_3^{(1)} \]
\[ + K_{64}^{(1)} v_4^{(1)} + K_{65}^{(1)} v_5^{(1)} + K_{66}^{(1)} v_6^{(1)} + F_4^{(1)} \]
\[ + K_{31}^{(2)} v_1^{(2)} + K_{32}^{(2)} v_2^{(2)} + K_{33}^{(2)} v_3^{(2)} \]
\[ + K_{34}^{(2)} v_4^{(2)} + K_{35}^{(2)} v_5^{(2)} + K_{36}^{(2)} v_6^{(2)} + F_1^{(2)} \]  

where

Member Stiffness Relations

(Constitutive Equations)

Since the end forces \( F_{1m} \) in (17.30) are unknown, the member stiffness relations

\[ \begin{bmatrix} F_1 \end{bmatrix}^{(m)} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} + \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \\ F_{15} \\ F_{16} \end{bmatrix} \]  
(17.31)

Compatibility

Imposing the compatibility (continuity) conditions

\[ v_1^{(1)} = v_2^{(1)} = v_3^{(1)} = 0 \]  
(17.35)

\[ v_4^{(1)} = d_1; \; v_5^{(1)} = d_2; \; v_6^{(1)} = d_3 \]

\[ v_1^{(2)} = d_1; \; v_2^{(2)} = d_2; \; v_3^{(2)} = d_3 \]  
(17.36)

on the constitutive equation version of the equilibrium equations leads to

\[ P_1 = (K_{44}^{(1)} + K_{11}^{(2)}) d_1 + (K_{45}^{(1)} + K_{12}^{(2)}) d_2 \]
\[ + (K_{46}^{(1)} + K_{13}^{(2)}) d_3 + (F_4^{(1)} + F_1^{(2)}) \]  
(17.39a)
The structure stiffness coefficients \( S_{ij} \) are defined in the usual manner, i.e., the force at dof \( i \) due to a unit displacement at \( j \) with all other displacements \( d_k = 0 \) and \( k \neq j \).

Assembly of \([S]\) and \(\{P_f\}\) Using Member Code Numbers

The explicit details given for the direct stiffness procedure highlights how the basic equations of structural analysis are utilized in matrix structural analysis. A disadvantage of the procedure is that it is tedious and not amenable to computer implementation.

Computer implementation is based on using a “destination array”

\[
\text{ID}(\text{NNDF}, \text{NNP}) \quad (\text{NNDF} \equiv \text{Number of Nodal Degrees of Freedom, three} \\
\text{NNP} \equiv \text{Number of Node Points, i.e., number of nodes used in the 2D structure discretization). The ID( , ) array identifies the nodal equation numbers in sequence:}
\]

\[
\begin{align*}
\text{ID}(1, \text{node}) &= \text{X-axis nodal displ. number} \\
\text{ID}(2, \text{node}) &= \text{Y-axis nodal displ. number} \\
\text{ID}(3, \text{node}) &= \text{Nodal rotation displ. number}
\end{align*}
\]

For example, consider the gable frame structure shown on the next slide. The destination array is also shown on this slide.
The ID( , ) array along with the element node number array IEN(NEN, NEL) (NEN = Number of Element Nodes, which is two for discrete structural elements; and NEL = Number of Elements used to discretize the structure) is used to construct the element location matrix array LM(NEDF, NEL) (NEDF = Number of Element Degrees of Freedom, which is six for the 2D frame elements), i.e.,

LM(i, iel) = ID(i, IEN(1,iel)); 1 = b node
LM(j, iel) = ID(i, IEN(2,iel)); 2 = e node
iel = element number

The IEN( , ) and LM( , ) arrays for the gable frame example are

<table>
<thead>
<tr>
<th>Node</th>
<th>ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 0 0</td>
</tr>
<tr>
<td>2</td>
<td>1 2 3</td>
</tr>
<tr>
<td>3</td>
<td>4 5 6</td>
</tr>
<tr>
<td>4</td>
<td>7 8 9</td>
</tr>
<tr>
<td>5</td>
<td>0 0 10</td>
</tr>
</tbody>
</table>

LM(iel, iel) = [K_{11} K_{12} K_{13} K_{14} K_{15} K_{16}]
LM(2, iel) = [K_{21} K_{22} K_{23} K_{24} K_{25} K_{26}]
LM(3, iel) = [K_{31} K_{32} K_{33} K_{34} K_{35} K_{36}]
LM(4, iel) = [K_{41} K_{42} K_{43} K_{44} K_{45} K_{46}]
LM(5, iel) = [K_{51} K_{52} K_{53} K_{54} K_{55} K_{56}]
LM(6, iel) = [K_{61} K_{62} K_{63} K_{64} K_{65} K_{66}]

The LM( , ) array is used to assemble the global element stiffness matrix and fixed-end force vector as illustrated on the next two slides.
For the example gable frame:

\[ S_{11} \leftarrow K^{(1)}_{44} + K^{(2)}_{11} \]
\[ S_{21} \leftarrow K^{(1)}_{54} + K^{(2)}_{21} \]
\[ \vdots \]
\[ S_{9,9} \leftarrow K^{(3)}_{66} + K^{(4)}_{66} \]
\[ S_{10,9} \leftarrow K^{(4)}_{36} \]
\[ S_{10,10} \leftarrow K^{(4)}_{33} \]

\[ P_{f1} \leftarrow F^{(1)}_{f1} + F^{(2)}_{f1} \]
\[ P_{f2} \leftarrow F^{(1)}_{f2} + F^{(2)}_{f2} \]
\[ \vdots \]
\[ P_{f9} \leftarrow F^{(3)}_{f6} + F^{(4)}_{f6} \]
\[ P_{f10} \leftarrow F^{(4)}_{f3} \]

SUMMARY

- Identify structure degrees of freedom \( d \)
- For each element:
  - Evaluate \([k]\), \(\{Q_f\}\), and \([T]\)
  - Calculate \([K] = [T]^T [k] [T]\)
  - \((F_j) = [T]^T (Q_j)\)
- Assemble \([K]\) into \([S]\)
- Form nodal load vector \(\{P\}\)
- Solve: \([S]\) \(\{d\}\) = \(\{P\} - \{P_f\}\)
- For each element:
  - Obtain \(\{v\}\) from \(\{d\}\) (compatibility)
  - Calculate \(\{u\} = [T]\{v\}\)
  - \(\{Q\} = [k]\{u\} + \{Q_f\}\)
- Determine support reactions by considering support joint equilibrium

Self-Straining: structure that is internally strained and in a state of stress while at rest without sustaining any external loading

Example Problems

- Truss
- Frame
**Some Features of the Stiffness Equations**

An important characteristic of both element and global (structure) linear elastic stiffness equations is that they are symmetric. Practically, this means that only the main diagonal terms and coefficients to one side of the main diagonal need to be generated and stored in a computer program. Also note that the stiffness (equilibrium) equation for a given degree of freedom is influenced by the degrees of freedom associated with the elements connecting to that degree of freedom. Thus, nonzero coefficients in a given row of a stiffness matrix consist only of the main diagonal and coefficients corresponding to degrees of freedom at that node and at other nodes on the elements meeting at the node for which the stiffness equation is being formed.
All other coefficients in the row are zero. When there are many degrees of freedom in the complete structure, the stiffness matrix may contain relatively few nonzero terms, in which case it is characterized as \textit{sparse or weakly populated}.

\textit{Banded Stiffness Equations}

Clearly, it would be advantageous in the solution phase of analysis to cluster all nonzero coefficients close to the main diagonal. This isolation of the zero terms facilitates their removal in the solution process. This can be done by numbering the degrees of freedom in such a way that the columnar distance of the term most remote from the main diagonal coefficient in each row is minimized; i.e., by minimizing the \textit{bandwidth}. This is most easily achieved by minimizing the nodal difference between connected elements. All commercial structural and finite element codes have built in schemes to minimize the interaction of the stiffness equations regardless of the user-specified input.

\textbf{Indeterminacy}

Cognizance of \textit{static indeterminacy} is not necessary in the direct stiffness approach. The displacement approach uses the analogous concept of \textit{kinematic indeterminacy}, refers to the number of displacement degrees of freedom that are required to define the displacement response of the structure to any load. Stated another way, \textit{kinematic indeterminacy equals the...
number of displacement degrees of freedom that must be constrained at the nodes in addition to the boundary constraints (support conditions) to reduce the system to one in which all the nodal displacements are zero or have predetermined values, e.g., specified support settlement.

It is fairly obvious that kinematic indeterminacy is quite different from static indeterminacy.