Stability Analysis of Conewise Affine Dynamical Systems Using Conewise Linear Lyapunov Functions

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***Abstract*— This paper proposes computational algorithms for analyzing conewise affine dynamical systems, where every neighborhood of the origin contains an affine mode. These algorithms are based on conewise linear Lyapunov functions. To make such algorithms useful, we present an algorithm to automatically search over partitions defining these conewise Linear functions. This algorithm is sound, although we present a counter-example to its completeness. We show that this approach verifies stability of 2D and 3D examples of conewise affine dynamical systems, including combinations of the harmonic and nonsmooth oscillators.**

***Index Terms*— Computational methods, Lyapunov methods, Optimization, Switched Systems.**

I. Introduction

Stability analysis for switched and hybrid dynamical systems has applications in many areas [1], [2]. Such systems often focus on affine dynamics or affine differential inclusions in each mode, since piecewise affine functions may suitably approximate a broad range of functions [3]–[5]. The analysis of systems where the dynamics in each mode are affine often uses common [1], [6], multiple [7], or piecewise [8]–[11] quadratic Lyapunov functions. More often, however, such Lyapunov functions are used to design switching rules between linear or affine dynamics [12]–[15], instead of analyzing a *given* switched or hybrid system.

Piecewise linear (PWL) Lyapunov functions were studied as an alternative to piecewise quadratic (PWQ) functions in [16] and [9]. Despite some theoretical [16] and computational advantages [9] of PWL functions over PWQ ones, the latter still dominate stability analysis.

Related Work And Motivation: To the best of our knowledge, algorithms for *analysis* of dynamical systems with affine dynamics in a neighborhood of the origin do not exist. This statement does not contradict existence of algorithms that *synthesize* switching rules between affine modes [12], [17], [18]. Johansson uses technical conditions in [9] that are unable to deal with affine dynamics in regions neighboring the origin (see Remark 1); Baier *et al.* do the same in [19]. Oehlerking *et al.* mention piecewise affine dynamics, however the example provided only contains piecewise linear dynamics [10]. The work in [20] is restricted to conewise linear differential inclusions [21] using polytopic Lyapunov functions [16]. Automated analysis of systems with piecewise constant (possibly set-valued) dynamics [22] – a subset of piecewise affine differential inclusions – have also been developed [23].

The ability to convert Lyapunov-based analysis of piecewise affine (PWA) dynamical systems into convex optimization problems has motivated attempts to automate the search for PWQ and PWL Lyapunov functions [9]–[11], [20]. These works observe that an automated partitioning scheme may enable analysis of systems for which using the same partition as the dynamics for the Lyapunov function fails to produce a valid Lyapunov function. The approaches in [9], [10] use the optimal dual variables from their optimization-based algorithm to identify which cell to refine. Iervolino *et al.* [11] use a vertex-edge representation of the partition, and insert a new vertex between the longest edge. All three approaches amount to splitting a common facet of adjacent cells. Poonawala [20] proposes new criteria for choosing which cells to refine, without solving dual optimization problems, and exact splitting techniques that go beyond splitting cells into equal volumes.

Contribution: This work proposes a refinement-based search algorithm for conewise linear Lyapunov functions that verify properties of conewise affine dynamical systems. These properties may include stability and asymptotic stability. This paper therefore extends [20] by considering a larger set of properties, Lyapunov functions, and dynamics. We use conditions for checking sign-definiteness of affine functions first presented in [17]. The work in [17] focuses on set invariance, and does not propose a refinement-based search algorithm for Lyapunov functions.

II. Background

Notation: The indices of the elements of a set S form the set $I(S)$. The symbol $\mathbf{x}_S$ denotes a set of variables $\{x_i\}_{i\in I(S)}$, and $\mathbf{x}_I$ to denote a set of variables $\{x_i\}_{i\in I}$. We denote the convex hull of a set S by $\operatorname{conv}(S)$, the interior of S by $\operatorname{Int}(S)$, the boundary of S by ∂S, and closure of S by $\overline{S}$.

The vector $\mathbf{1}_n \in \mathbb{R}^n$ has all elements equal to unity. We omit the subscript n if its value is clear from the context. We use E^i to denote the i^{th} row of matrix E, and $E^{i:j}$ to denote a matrix formed by rows i to j of E.

For $v, u \in \mathbb{R}^n$, $v \succeq u \iff v_i \geq u_i$ for $1 \leq i \leq n$. The symbols $\preceq$, $\succ$, and $\prec$ imply the same element-wise rule corresponding to $\leq$, $>$, and $<$ respectively.

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A. Partitions And Refinements

A partition $\mathcal{P}$ is a collection of subsets $\{X_i\}_{i\in I(\mathcal{P})}$, where $X_i \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, and X_i is regular closed ($\overline{\mathrm{Int}\,(X_i)} = X_i$) for each $i \in I(\mathcal{P})$. Furthermore, $\mathrm{Int}\,(X_i) \cap \mathrm{Int}\,(X_j) = \emptyset$ for each pair $i, j \in I(\mathcal{P})$ such that $i \neq j$. We refer to $\cup_{i\in I(\mathcal{P})} X_i$ as the domain of $\mathcal{P}$, which we also denote by $\mathrm{Dom}(\mathcal{P})$. We also refer to the subsets X_i in $\mathcal{P}$ as the cells of the partition. We assume that there exists a neighborhood of x that intersects with only a finite number of cells in $\mathcal{P}$, for each $x \in \mathrm{Dom}(\mathcal{P})$.

Let $\mathcal{P} = \{Y_i\}_{i\in I}$ and $\mathcal{R} = \{Z_j\}_{j\in J}$ be two partitions of a set $S = \mathrm{Dom}\,(\mathcal{P}) = \mathrm{Dom}\,(\mathcal{R})$. A partition $\mathcal{R}$ is a *refinement* of $\mathcal{P}$ if $Z_j \cap Y_i \neq \emptyset$ implies that $Z_j \subseteq Y_i$. We denote the set of refinements of a partition $\mathcal{P}$ as $\mathrm{Ref}(\mathcal{P})$. There exists a natural abstraction function $\pi_{\mathcal{R},\mathcal{P}} \colon I_{\mathcal{R}} \mapsto I_{\mathcal{P}}$, given by $\pi_{\mathcal{R},\mathcal{P}}(j) = \{i \in I_{\mathcal{P}} \colon Z_j \subseteq Y_i\}$.

B. Conewise Affine Dynamical Systems

A conewise affine (CWA) dynamical system $\Omega_{\mathcal{P}}$ associated with partition $\mathcal{P} = \{X_j\}_{j\in I(\mathcal{P})}$ is a collection,

$$\Omega_{\mathcal{P}} = \{A_j x + a_j\}_{j\in I(\mathcal{P})} \tag{1}$$

that to each cell $X_j \in \mathcal{P}$ assigns affine dynamics. Therefore,

$$\dot{x}(t) = A_j x(t) + a_j, \text{ if } x(t) \in X_j, \text{ where} \tag{2}$$
$$X_j = \{x \in \mathbb{R}^n \colon F_j x + f_j \succeq 0\} \quad \forall j \in I(\mathcal{P}). \tag{3}$$

We require that either a) $F_j \in \mathbb{R}^{n\times n}$ and $f_j = 0$, or b) $F_j \in \mathbb{R}^{(n+1)\times n}$, F_j^{n+1} belongs to the interior of the negative polar cone [20] of $\{x \in \mathbb{R}^n \colon F_j^{1:n} x \succeq 0\}$, and $f_j \succeq 0$ with only the $(n+1)^{\text{th}}$ element being non-zero. In both cases, $F_j^{1:n}$ is full-rank. Therefore, X_i is either an n-sided unbounded cone or an $(n+1)$-sided polytope, with one vertex at the origin in both cases. Finally, we assume that $0 \in \mathrm{conv}(\{a_j\}_{j\in I(\mathcal{P})})$.

C. Conewise Linear Lyapunov Functions

We parameterize a conewise linear (CWL) Lyapunov function [9], [16], denoted by $V_{\mathcal{Q}}(x)$, by a partition $\mathcal{Q} = \{Z_i\}_{i\in I(\mathcal{Q})}$ and a collection of vectors $\mathbf{p}_{\mathcal{Q}} = \{p_i\}_{i\in I(\mathcal{Q})}$ such that

$$V_{\mathcal{Q}}(x) = p_i^T x, \text{ if } x \in Z_i, \text{ where} \tag{4}$$
$$Z_i = \{x \in \mathbb{R}^n \colon E_i x + e_i \succeq 0\} \tag{5}$$

where E_i, e_i satisfy identical conditions as in Section II-B.

We characterize adjacent sets in $\mathcal{Q}$ using the index set

$$I_{cont}(\mathcal{Q}) = \{(i,j) \in I(\mathcal{Q}) \times I(\mathcal{Q}) \colon Z_i \cap Z_j \neq \emptyset\}. \tag{6}$$

The non-empty boundary between two cells Z_i and Z_j in $\mathcal{Q}$ is parameterized by a vector η_{ij}, so that

$$Z_i \cap Z_j \subset \{x \in \mathbb{R}^n \colon \eta_{ij}^T x = 0\}. \tag{7}$$

The following results establish constraints on the parameters p_j and E_j for $j \in I(\mathcal{Q})$ that ensure $V_{\mathcal{Q}}(x)$ is a useful candidate Lyapunov function.

Lemma 1 (Lemma 4.7 [9])**.** *The following are equivalent*

1) $Ex \succeq 0, Ex \neq 0 \implies p^T x > 0.$
2) $\exists v \succ 0$ *such that* $E^T v = p$.

Lemma 2. *Consider a function $V_{\mathcal{Q}}(x)$ as defined in equations (4)-(7). If there exist variables μ_i for $i \in I(\mathcal{Q})$, λ_{ij} for $(i,j) \in I_{cont}(\mathcal{Q})$, and $\epsilon > 0$ that satisfy*

$$p_i = E_i^T \mu_i, \quad \forall i \in I(\mathcal{Q}), \tag{8}$$
$$\mu_i \succeq \epsilon \mathbf{1}, \quad \forall i \in I(\mathcal{Q}), \text{ and} \tag{9}$$
$$p_i - p_j = \lambda_{ij}\eta_{ij}, \quad \forall (i,j) \in I_{cont}(\mathcal{Q}), \tag{10}$$

then $V_{\mathcal{Q}}(x)$ is positive definite and locally Lipschitz.

Proof. $V_{\mathcal{Q}}(x)$ is piecewise linear. Assume that variables satisfying (8)-(10) exist. When $x \in Z_i \cap Z_j$, then $\eta_{ij}^T x = 0$ by definition. Therefore, condition (10) implies that $V_{\mathcal{Q}}(x)$ is continuous at its boundaries, so that $V_{\mathcal{Q}}(x)$ is locally Lipschitz. By construction, $V_{\mathcal{Q}}(0) = 0$. By Lemma 1, if conditions (8) and (9) hold, then $V_{\mathcal{Q}}(x) > 0$ when $x \neq 0$. Therefore, $V_{\mathcal{Q}}(x)$ is positive definite. ■

III. Problem Formulation

This paper deals with the following problem.

Problem 1. *Given a conewise affine dynamical system $\Omega_{\mathcal{P}}$, find a conewise linear Lyapunov function $V_{\mathcal{Q}}(x)$ that certifies one of the following properties:*

a) The origin of $\Omega_{\mathcal{P}}$ is stable.
b) The origin of $\Omega_{\mathcal{P}}$ is asymptotically stable.

IV. Lyapunov-Based Stability Conditions

This section derives conditions on the parameters of a CWA system $\Omega_{\mathcal{P}}$ and CWL candidate Lyapunov function $V_{\mathcal{Q}}(x)$ that ensure that $V_{\mathcal{Q}}(x)$ either decreases or is non-increasing along solutions of $\Omega_{\mathcal{P}}$. These conditions will be checked using an optimization problem.

A. Nonsmooth Analysis

When $V_{\mathcal{Q}}(x)$ is differentiable, and the dynamics $\dot{x} = f(x)$ are continuous, stability may be assessed through the condition that the Lie derivative $\mathcal{L}_f V_{\mathcal{Q}}(x)$ of $V_{\mathcal{Q}}(x)$ along $f(x)$ be negative definite or negative semi-definite for all x in some region $S \ni 0$:

$$\mathcal{L}_f V_{\mathcal{Q}}(x) = \langle \nabla V_{\mathcal{Q}}(x), f(x)\rangle < 0 \quad (\text{or} \leq 0), \tag{11}$$

where $\nabla V_{\mathcal{Q}}(x)$ is the gradient of $V_{\mathcal{Q}}(x)$ and $\langle \cdot, \cdot \rangle$ is the usual inner product.

Since we focus on CWL Lyapunov functions $V_{\mathcal{Q}}(x)$ and CWA dynamics $\Omega_{\mathcal{P}}$, condition (11) becomes

$$p_i^T (A_j x + a_j) < 0 \quad (\text{or} \leq 0), \tag{12}$$

when $x \in \mathrm{Int}\,(X_j) \cap \mathrm{Int}\,(Z_i)$, where $i \in I(\mathcal{Q})$ and $j \in I(\mathcal{P})$. When x belongs to the boundary of cells in $V_{\mathcal{Q}}(x)$ (where it is non-differentiable) or the boundary of cells in $\Omega_{\mathcal{P}}$ (where dynamics are discontinuous), the quantities $\nabla V_{\mathcal{Q}}(x)$ and $f(x)$ must be replaced by their multi-valued extensions [24], [25]. The multi-valued nature of these quantities implies that we may need multiple conditions of the form (12) to hold at such boundaries.

Due to the piecewise nature of $V_{\mathcal{Q}}(x)$ and $\Omega_{\mathcal{P}}$, the same set of conditions of the form (12) may be simultaneously checked for all points x satisfying conditions of the form $Gx+g \succeq 0$, instead of for all $x \in \mathbb{R}^n$. We handle the relevant combinations of i, j, and such sets as follows. Consider a relation R on X given by

$$\begin{aligned} &x_1 R x_2 \implies I_{\mathcal{P}}(x_1) = I_{\mathcal{P}}(x_2), I_{\mathcal{Q}}(x_1) = I_{\mathcal{Q}}(x_2), \text{ where} \\ &I_{\mathcal{P}}(x) = \{j \in I(\mathcal{P}) \colon x \in X_j\}, \text{and} \\ &I_{\mathcal{Q}}(x) = \{i \in I(\mathcal{Q}) \colon x \in Z_i\}. \end{aligned}$$

This relation, based on membership in $\mathcal{Q}$ and $\mathcal{P}$, is an equivalence relation. Let $\mathcal{G}(\mathcal{Q},\mathcal{P}) = \{Y_k\}_{k \in I(\mathcal{G})}$ be the set of equivalence classes of X under R, where

$$\overline{Y_k} = \{x \in \mathbb{R}^n \colon G_k x + g_k \succeq 0\}. \tag{13}$$

This construction makes $\mathcal{G}(\mathcal{Q},\mathcal{P})$ a simplical complex; it is not, however, a partition in our sense. The open cells of $\mathcal{G}(\mathcal{Q},\mathcal{P})$ correspond to regions where $V_{\mathcal{Q}}(x)$ is differentiable and the dynamics are continuous; the remaining cells are regions of discontinuity and/or non-differentiability. Let

$$I_{dec}(\mathcal{Q},\mathcal{P}) = \{(i,j,k) \colon (i,j) \in I_{\mathcal{Q}}(x) \times I_{\mathcal{P}}(x) \text{ for } x \in Y_k, k \in \kappa(\mathcal{G})\}, \text{ where,} \tag{14}$$

$$\kappa(\mathcal{G}) = \{k \in I(\mathcal{G}) \colon \text{ dynamics are either single-valued in } Y_k, \text{ or sliding occurs in } Y_k\}. \tag{15}$$

The set $I_{dec}(\mathcal{Q},\mathcal{P})$ will pick out the right conditions on parameters of $V_{\mathcal{Q}}(x)$ and $\Omega_{\mathcal{P}}$ required to establish strict decrease or non-increase of $V_{\mathcal{Q}}(x)$. We also define subsets

$$\begin{aligned} &I^l_{dec}(\mathcal{Q},\mathcal{P}) = \{(i,j,k) \in I_{dec}(\mathcal{Q},\mathcal{P}) \colon g_k = 0 \text{ and } a_j = 0\}, \\ &I^a_{dec}(\mathcal{Q},\mathcal{P}) = I_{dec}(\mathcal{Q},\mathcal{P}) \backslash I^l_{dec}(\mathcal{Q},\mathcal{P}) \end{aligned}$$

To convert the conditional statement $Gx + g \succeq 0 \implies p^T x + q \leq 0$ into a constraint without conditions, we use the following result shown in [17]:

Lemma 3 ([17]). *Let the set $\{x \in \mathbb{R}^n \colon Gx + g \succeq 0\}$ be non-empty, where $G \in \mathbb{R}^{l \times n}$, $g \in \mathbb{R}^l$ for some $l \in \mathbb{N}$. Let $p \in \mathbb{R}^n$ and $q \in \mathbb{R}$. Then, the following are equivalent*

1) $Gx + g \succeq 0 \implies p^T x + q \leq 0$.
2) $\exists v \in \mathbb{R}^l$, $v \succeq 0$ *such that* $G^T v + p = 0$ *and* $g^T v + q \leq 0$.

Remark 1. Lemma 3 allows us to exactly analyze affine dynamics in sets that include the origin on their boundary, unlike the approach in [9] based on Lemma 1.

In summary, given function $V_{\mathcal{Q}}(x)$ consisting of a partition $\mathcal{Q}$ and corresponding parameters $\mathbf{p}_{\mathcal{Q}}$, and a PWA dynamical system $\Omega_{\mathcal{P}}$, we may formulate a set of constraints on $\mathbf{p}_{\mathcal{Q}}$. These constraints lead to an optimization problem that implements a search for a CWL Lyapunov function that verifies properties of $\Omega_{\mathcal{P}}$. The next section describes this optimization problem.

B. Optimization-Based Nonsmooth Lyapunov Analysis

Consider an objective function $J_{\Omega_{\mathcal{P}},\mathcal{Q}}(\mathbf{w})$ given by

$$J_{\Omega_{\mathcal{P}},\mathcal{Q}}(\mathbf{w}) = \sum_{(i,j,k) \in I_{dec}(\mathcal{Q})} w^d_{ijk} \|s_{ijk}\| + w^c_{ijk} \|t_{ijk}\|, \tag{16}$$

where $\mathbf{w}$ represents the set of non-negative weight parameters w^d_{ijk} and w^c_{ijk} for $(i,j,k) \in I_{dec}(\mathcal{Q},\mathcal{P})$, and s_{ijk}, t_{ijk} are variables we introduce below. The optimization problem $\text{Opt}(\Omega_{\mathcal{P}},\mathcal{Q},\mathbf{w})$ that implements a search for $\mathbf{p}_{\mathcal{Q}}$ given $\mathcal{Q}$ is

$$\min_{p_i, \mu_i, \nu_{ijk}, \lambda_{ij}, s_{ijk}, t_{ijk}} \quad J_{\Omega_{\mathcal{P}},\mathcal{Q}}(\mathbf{w}) \tag{17}$$

$$\text{s.t.} \quad p_i = \left(E^{1:n}_i\right)^T \mu_i, \quad \forall i \in I(\mathcal{Q}), \tag{18}$$

$$\mu_i \succeq \epsilon_1 \mathbf{1}, \quad \forall i \in I(\mathcal{Q}), \tag{19}$$

$$p_i - p_j = \lambda_{ij} \eta_{ij}, \quad \forall (i,j) \in I_{cont}(\mathcal{Q}), \tag{20}$$

$$s_{ijk} = G^T_k \nu_{ijk} + A^T_j p_i, \quad \forall (i,j,k) \in I^l_{dec}(\mathcal{Q},\mathcal{P}), \tag{21}$$

$$\nu_{ijk} \succeq \epsilon_2 \mathbf{1}, \quad \forall (i,j,k) \in I^l_{dec}(\mathcal{Q},\mathcal{P}), \tag{22}$$

$$s_{ijk} = G^T_k \nu_{ijk} + A^T_j p_i, \quad \forall (i,j,k) \in I^a_{dec}(\mathcal{Q},\mathcal{P}), \tag{23}$$

$$t_{ijk} \geq \epsilon_2 + g^T_k \nu_{ijk} + a^T_j p_i, \quad \forall (i,j,k) \in I^a_{dec}(\mathcal{Q},\mathcal{P}), \tag{24}$$

$$\nu_{ijk} \succeq 0, \quad t_{ijk} \geq 0, \quad \forall (i,j,k) \in I^a_{dec}(\mathcal{Q},\mathcal{P}), \tag{25}$$

where $\epsilon_1 > 0$ and $\epsilon_2 \geq 0$. The optimization problem $\text{Opt}(\Omega_{\mathcal{P}},\mathcal{Q},\mathbf{w})$ in (17)-(25) contains several variables, of which $\mathbf{p}_{\mathcal{Q}}$ is most important, and the rest are related to establishing properties of the function $V_{\mathcal{Q}}(x)$. We state the following result:

Lemma 4. *The optimization problem* $\text{Opt}(\Omega_{\mathcal{P}},\mathcal{Q},\mathbf{w})$ *in* (17)-(25) *is always feasible given the definitions of* $V_{\mathcal{Q}}(x)$ *and* $\Omega_{\mathcal{P}}$.

Proof. This result is by construction. By constraining $\mathcal{Q}$ to have apex at the origin, we may always find a continuous positive definite function $V_{\mathcal{Q}}(x)$. Thus, constraints (18)-(20) are always feasible by themselves. The remaining constraints are always feasible for any value of $\mathbf{p}_{\mathcal{Q}}$ due to use of slack-like variables s_{ijk} and t_{ijk}. Therefore $\text{Opt}(\Omega_{\mathcal{P}},\mathcal{Q},\mathbf{w})$ is always feasible. ■

While $\text{Opt}(\Omega_{\mathcal{P}},\mathcal{Q},\mathbf{w})$ is always feasible, only the case where the optimal value is zero is useful. The following result makes this statement precise.

Theorem 5. *Let the optimal value of* $\text{Opt}(\Omega_{\mathcal{P}},\mathcal{Q},\mathbf{w})$ *be zero for a given partition* $\mathcal{Q}$ *and system* $\Omega_{\mathcal{P}}$. *If* $\epsilon_2 > 0$, *then the origin is (strongly) asymptotically stable, otherwise the origin is (strongly) stable*

Remark 2. The term *strong* implies that the conclusion applies to all solutions of the nonsmooth dynamics [26], which are potentially not unique for some initial conditions.

Proof. Constraints (18)-(20) ensure that $V_{\mathcal{Q}}(x)$ is a valid candidate Lyapunov function, by Lemma 2. Furthermore, it is a non-pathological function [27] since it is conewise linear. We will show that the derivative $\frac{\mathrm{d}}{\mathrm{d}t} V_{\mathcal{Q}}(\phi(t))$ along a solution $\phi(t)$ of $\Omega_{\mathcal{P}}$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t} V_{\mathcal{Q}}(\phi(t)) \leq 0$ (or < 0) almost everywhere.

First, we can exclude times when the solutions passes through a discontinuity of the dynamics without sliding, whether or not $V_{\mathcal{Q}}(x)$ is differentiable there. These times occur when $\phi(t) \in Y_k$, $k \notin \kappa(\mathcal{G})$, by construction of $\mathcal{G}$. If $V_{\mathcal{Q}}(x)$ is differentiable at such x, we may exclude these time instants by arguments in [25] (page 156). If $V_{\mathcal{Q}}(x)$ is non-differentiable, the set-valued Lie derivative $\dot{\overline{V}}(x)$ [26], [28] turns out to always be empty for such x, since $V_{\mathcal{Q}}(x)$ is conewise linear. Therefore, by Proposition 2 in [28], we may ignore these times.

We must now address $\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathcal{Q}}(\phi(t))$ when $\phi(t) \in Y_k$ where $k \in \kappa(\mathcal{G})$. If the solution of $\mathrm{Opt}(\Omega_{\mathcal{P}}, \mathcal{Q}, \mathbf{w})$ is such that $s_{ijk} = 0$ and $t_{ijk} = 0$ for all $(i,j,k) \in I_{dec}(\mathcal{Q}, \mathcal{P})$, then by Lemmas 1 and 3, we may conclude that $p_i^T(A_j^T x + a_j) \leq -\epsilon_3$ when $x \in Y_k$, for each $(i,j,k) \in I_{dec}(\mathcal{Q}, \mathcal{P})$, where $\epsilon_3 > 0$ if $\epsilon_2 > 0$, and $\epsilon_3 = 0$ if $\epsilon_2 = 0$. In turn, $p^T f \leq -\epsilon_3$, where $p \in \mathrm{conv}\left(\{p_i\}_{i \in I_{\mathcal{Q}}(x)}\right)$ and $f \in \mathrm{conv}\left(\{A_j x + a_j\}_{j \in I_{\mathcal{P}}(x)}\right)$ ([19], Proposition 3.6). Due to the properties of the set-valued Lie derivative $\dot{\overline{V}}(x)$ and properties of $V_{\mathcal{Q}}(x)$, we may conclude ([28], Proposition 2) that $\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathcal{Q}}(\phi(t)) \leq -\epsilon_3$ when $\phi(t) \in Y_k$ where $k \in \kappa(\mathcal{G})$. Therefore, we have established decrease (or non-increase) of $V_{\mathcal{Q}}(x)$ almost everywhere along solutions of $\Omega_{\mathcal{P}}$. By Theorems 1 and 3 in [26], the result is proved. ■

In summary, we may solve the optimization problem $\mathrm{Opt}(\Omega_{\mathcal{P}}, \mathcal{Q}, \mathbf{w})$ for a given partition $\mathcal{Q}$ and unknown parameters $\mathbf{p}_{\mathcal{Q}}$ and potentially verify stability properties of the origin. What happens when the optimal value of $\mathrm{Opt}(\Omega_{\mathcal{P}}, \mathcal{Q}, \mathbf{w})$ is non-zero? Since Lemmas 1 and 3 involve exact alternatives, we may conclude that no valid CWL Lyapunov functions exists corresponding to $\mathcal{Q}$. The next section describes how we continue searching for a valid CWL Lyapunov function in this situation.

V. A Search Algorithms For CWL Lyapunov Functions Via Sequential Refinement

This section proposes an algorithm for solving Problem 1 using sequential refinements. We consider algorithms that solve a sequence of optimization problems $\mathrm{Opt}(\Omega_{\mathcal{P}}, \mathcal{R}_m, \mathbf{w})$ of the form (17)-(25) corresponding to a sequence of partitions $\mathcal{R}_m$ where $\mathcal{R}_{m+1} \in \mathrm{Ref}(\mathcal{R}_m)$ and $\mathcal{R}_0 = \mathcal{P}$. The idea is to use variables s_{ijk} and t_{ijk} for $(i,j,k) \in I_{dec}(\mathcal{R}_m, \mathcal{P})$ with non-zero values to guide this refinement, until we arrive at a partition for which they are all zero, yielding a valid Lyapunov function. Designing this algorithm requires some choices to be made, leading to multiple possible algorithms. We describe one set of choices in the sections below, leading to our algorithm presented in Section V-D.

A. Constructing $\mathcal{G}$ and $\mathrm{I_{dec}}(\mathcal{R}_m, \mathcal{P})$

Since we choose $\mathcal{R}_0 = \mathcal{P}$, the cells in $\mathcal{G}$ consist of the cells in $\mathcal{R}_m$, in addition to boundaries of cells in $\mathcal{R}_m$ that coincide with boundaries in $\mathcal{P}$. For most cells Z_i in $\mathcal{G}$, we obtain only one condition of the form (12) corresponding to a i and $\{j\} = \pi_{\mathcal{R}_m, \mathcal{P}}(i)$, and include a single corresponding multi-index (i,j,k) in $I_{dec}(\mathcal{R}_m, \mathcal{P})$. For cells corresponding to sliding surfaces of $\Omega_{\mathcal{P}}$, we include indices corresponding to all combinations of dynamics $A_j x + a_j$ and linear parameters p_i corresponding to the cells neighboring the sliding surface.

B. Choosing Cells To Split

When the optimal value of $\mathrm{Opt}(\Omega_{\mathcal{P}}, \mathcal{R}_m, \mathbf{w})$ is non-zero, we refine the cells of $\mathcal{R}_m$. One strategy is to split all cells, however the size of the partition, and therefore $\mathrm{Opt}(\Omega_{\mathcal{P}}, \mathcal{R}_m, \mathbf{w})$, may grow too quickly, creating high computational demands.

Algorithm 1 Verifying Stability By Refining CWL Lyapunov Functions

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Require: $\Omega_{\mathcal{P}}$, $\mathbf{w}$, $\epsilon_1 > 0$, $\epsilon_2 \geq 0$
Ensure: PWL Lyapunov function $V_{\mathcal{Q}}(x)$ that verifies the origin is (asymptotically) stable.

```
m ← 0 {Loop counter}
R_m ← P, J_m ← ∞
while J_m ≠ 0 do
    Solve Opt(Ω_P, R_m, w).
    I_s ← {i ∈ I(R_m): s_ijk ≠ 0}
    I_t ← {i ∈ I(R_m): t_ijk ≠ 0}
    for i ∈ I_s do
        Refine Z_i ∈ R_m (see Section V-C)
    end for
    if I_s = ∅ then
        for i ∈ I_t do
            Refine Z_i ∈ R_m (see Section V-C)
        end for
    end if
    m ← m + 1
end while
Q ← R_m
return V_Q(x).
```

To choose which cells to refine, we first define two index sets I_s and I_t, where

$$I_s = \{i \in I(\mathcal{R}_m) \colon s_{ijk} \neq 0\}, \text{ and}$$
$$I_t = \{i \in I(\mathcal{R}_m) \colon t_{ijk} \neq 0\}.$$

We then propose the following method to choose which cells to refine.

1) If I_s is not empty, split all cells in I_s.
2) Otherwise, split all cells in I_t.

C. Splitting Cells

Each cell Z_i in $\mathcal{R}_m$ is a cone with n-facets defined by $E_i^{1:n}$ (see Section II-C). To each cell i we may associate variables s_{ijk} and t_{ijk} where $(i,j,k) \in I_{dec}(\mathcal{R}_m, \mathcal{P})$. Poonawala shows [20] that $\|s_{ijk}\|$ may be interpreted as a distance between $-A_j^T p_i$ and the dual cone of Z_i. If $s_{ijk} \neq 0$, then we compute $\nu_i = \left(\left(E_i^{1:n}\right)^T\right)^{-1} s_{ijk}$. We identify

$$j_+ = \arg\min_{i:\nu_i \geq 0} \nu_i, \text{ and} \tag{26}$$

$$j_- = \arg\max_{i:\nu_i < 0} \nu_i, \tag{27}$$

and use the hyperplane defined by row vector $0.5E_i^{j+} - 0.5E_i^{j-}$ to divide the cell, assuming j_+ and j_- exist. This hyperplane contains the origin. This split of Z_i into two cells will divide $n-2$ facets of Z_i. To ensure that all cells have n neighbors, we must split all cells in $\mathcal{R}_m$ that share one of these divided facets with Z_i. These additional splits are not required in 2 dimensions, one additional split is required in 3 dimensions, and more complicated combinations are required in higher dimensions. Without these additional splits, maintaining continuity of $V_{\mathcal{R}_m}(x)$ through (20) becomes difficult.

D. Our Refinement Algorithm

Algorithm 1 describes the algorithm resulting from the choices described in the previous sections. We can show the following properties.

Proposition 6. *Algorithm 1 is sound.*

Proof. The algorithm terminates when $\mathrm{Opt}(\Omega_{\mathcal{P}}, \mathcal{R}_m, \mathbf{w})$ has optimal value zero. By Theorem 5, the candidate Lyapunov function $V_{\mathcal{Q}}(x)$ is therefore a valid Lyapunov function certifying strong asymptotic stability of the origin if $\epsilon_2 > 0$, and strong stability if $\epsilon_2 = 0$. ∎

Proposition 7. *Algorithm 1 is not complete.*

Proof. Consider the simple harmonic oscillator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 \tag{28}$$

The origin is stable, and every disk centered at the origin is positively invariant. No polytope with finite number of sides containing the origin is positively invariant with respect to these dynamics (an application of Green's Theorem). In turn, no CWL function with a finite number of pieces can be non-increasing at all points. Therefore, Algorithm 1 will fail to verify that this system is stable. ∎

VI. Examples

We present four examples that demonstrate the performance of the refinement-based search for a CWL Lyapunov function proposed in Section V. We use the `Mosek` optimization package and `Julia v1.3` to implement all computations, using a computer with a 2.6 GHz processor and 16 GB RAM. The values of ϵ_1 and ϵ_2 are set to 1 and 0.001 respectively. We use a tolerance of 10^{-8} when checking if a number is non-zero. All weights in $\mathbf{w}$ are set to 1.

For $\mathbb{R}^2$, let X_1, X_2, X_3, and X_4 represent the four quadrants of $\mathbb{R}^2$, with $X_1 = \{(x_1, x_2)\colon x_1 \geq 0, x_2 \geq 0\}$, and the indices increasing counter-clockwise. For $\mathbb{R}^3$, let X_1, X_2, X_3, X_4 represent the four quadrants where $x_3 \geq 0$, and X_{i+4} be the quadrant corresponding to X_i reflected about the x_1, x_2-plane.

We summarize the performance of Algorithm 1 on the examples below in Table I. The computation time is low for these examples, however it is not hard to modify the parameters to need still greater number of cells, increasing the computation time. Examples 1 and 4 are designed for this paper.

Example 1 (Conewise affine 2D example)**.** The system dynamics are

$$\Omega_{\mathcal{P}}: \quad \dot{x} = \begin{cases} Ax + a_1, & \text{if } x \in X_1, \\ a_2, & \text{if } x \in X_2, \\ a_3, & \text{if } x \in X_3, \\ Ax + a_4, & \text{if } x \in X_4. \end{cases}$$

Example	Dynamics	No. of. Cells	Computation time	Verified property
1	CWA	144	0.655 sec	AS
2 [19]	CWA	4	0.242 sec	AS
3 [9]	CWL	256	1.25 sec	AS
4	CWL	20	0.74 sec	AS

TABLE I: Summary of examples. AS: Strong asymptotic stability.

where

$$A = \begin{bmatrix} -0 & 1 \\ -1 & -0 \end{bmatrix},$$
$$a_1 = \begin{bmatrix} 0.3 \\ -1 \end{bmatrix}, \ a_2 = \begin{bmatrix} 1 \\ 0.9 \end{bmatrix}, \ a_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ a_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Example 2 (Nonsmooth oscillator with nonsmooth friction [19])**.** The dynamics are given by

$$\dot{x} = \begin{bmatrix} -\mathrm{sign}\,(x_2) - \frac{1}{2}\mathrm{sign}\,(x_1) \\ \mathrm{sign}\,(x_1) \end{bmatrix}$$

These dynamics imply a conewise affine dynamical system $\Omega_{\mathcal{P}}$ where the $\mathcal{P}$ corresponds to the four quadrants of $\mathbb{R}^2$.

Example 3 (2D Example from [9], [28])**.** This example uses cells different from the four quadrants of $\mathbb{R}^2$. Consider cells

$$\begin{aligned} Z_1 &= \{x \in \mathbb{R}^2\colon -x_1 + x_2 \geq 0, x_1 + x_2 \geq 0\}, \\ Z_2 &= \{x \in \mathbb{R}^2\colon -x_1 + x_2 \geq 0, -x_1 - x_2 \geq 0\}, \\ Z_3 &= \{x \in \mathbb{R}^2\colon x_1 - x_2 \geq 0, -x_1 - x_2 \geq 0\}, \text{ and} \\ Z_4 &= \{x \in \mathbb{R}^2\colon x_1 - x_2 \geq 0, x_1 + x_2 \geq 0\}. \end{aligned}$$

Then,

$$\Omega_{\mathcal{P}}: \quad \dot{x} = \begin{cases} \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix} x \text{ if } x \in Z_1 \text{ or } x \in Z_3, \\ \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix} x \text{ if } x \in Z_2 \text{ or } x \in Z_4. \end{cases}$$

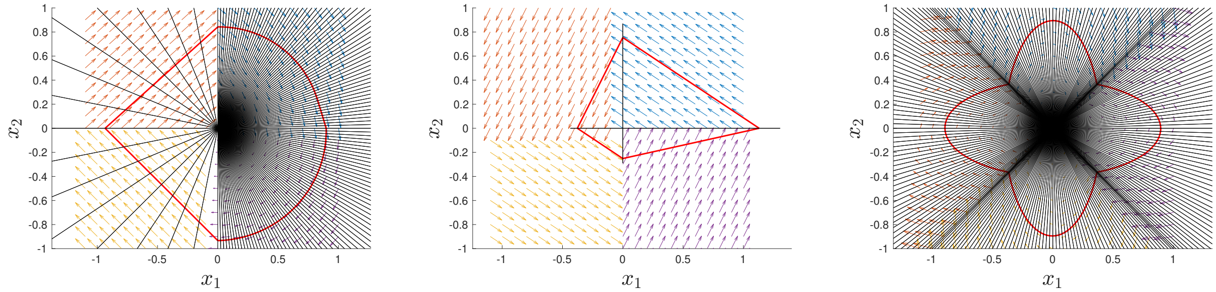
Example 4 (3D example based on [7])**.** Consider the following matrices derived from the classic example in [7]:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -0.1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4.0 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -0.1 & 0 \end{bmatrix}$$

Then, the conewise linear dynamics are

$$\Omega_{\mathcal{P}}: \quad \dot{x} = \begin{cases} (A_1 + B_1)x & \text{if } x \in X_1 \text{ or } x \in X_7 \\ (A_2 + B_1)x & \text{if } x \in X_2 \text{ or } x \in X_8 \\ (A_1 + B_2)x & \text{if } x \in X_3 \text{ or } x \in X_5 \\ (A_2 + B_2)x & \text{if } x \in X_4 \text{ or } x \in X_6 \end{cases}$$

These dynamics exhibit sliding on the boundary between X_1 and X_4, and X_6 and X_7.



(a) Example 1 (b) Example 2 (c) Example 3

Fig. 1: Level sets (red) of valid Lyapunov functions found by Algorithm 1 for the examples in Section VI. Each plot also shows the vector field, and partition (black lines). We omit a plot for Example 4 due to space constraints.

VII. Discussion And Future Work

This paper introduces an algorithmic framework to search for conewise linear (CWL) Lyapunov functions that verify properties of conewise affine (CWA) dynamical systems. An important idea is to refine partitions defining candidate Lyapunov functions, instead of manually fixing the partition. We show that the resulting algorithm is sound, but not complete, using the harmonic oscillator as a counter-example. However, we demonstrate that the algorithm still verifies stability properties of both novel examples and examples from the literature.

Future Work: This work presents two avenues for future work. The first involves proposing criteria and methods to split cells using half-planes that do no pass through the origin, enabling a search over piecewise affine Lyapunov functions. The second involves an analysis of the computational complexity of the algorithm, and conditions on the system data that determine when the proposed algorithm may be expected to find a valid Lyapunov function.

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