

EE522 NOTES: APERTURE ANTENNAS

The equivalent Principle

- Volume region: V , surface of V is S , the outward normal direction of S is \hat{n} .
- Original problem: Source in V , Field generated by the source is E and H .
- General equivalent problem: Field internal to V is any field \bar{E}_1, \bar{H}_1 , Equivalent magnetic current M and electric current J on S . If the equivalent currents satisfy

$$\bar{J}_{S1} = \hat{n} \times (\bar{H} - \bar{H}_1), \quad \bar{M}_{S1} = (\bar{E} - \bar{E}_1) \times \hat{n}$$

Then the field generated by the equivalent source external to V are the same as the field in the original configuration.

- There is no restriction of the internal field \bar{E}_1, \bar{H}_1 in the above equivalent problem. Hence one choice is to make $\bar{E}_1 = 0, \bar{H}_1 = 0$. For this case, the equivalent currents are:

$$\bar{J}_S = \hat{n} \times \bar{H} \quad \bar{M}_S = \bar{E} \times \hat{n}, \quad (\text{on } S)$$

Since the field inside S is zero in this case, we are free to introduce any material to S (1. Same as external, 2. PEC, 3. PMC). For planar surface S , the problem can be simplified using PEC or PMC material inside S .

Field radiated by Js and Ms: is equal to the summation of the field radiated by Js and that by Ms:

$$\bar{E}_A = -j\omega(A_\theta \hat{\theta} + A_\phi \hat{\phi}), \quad \bar{A} = \mu \frac{e^{-j\beta r}}{4\pi r} \int_S \bar{J}_S(\bar{r}') e^{j\beta \hat{r} \cdot \bar{r}'} dS'$$

$$\bar{H}_F = -j\omega(F_\theta \hat{\theta} + F_\phi \hat{\phi}), \quad \bar{F} = \varepsilon \frac{e^{-j\beta r}}{4\pi r} \int_S \bar{M}_S(\bar{r}') e^{j\beta \hat{r} \cdot \bar{r}'} dS'$$

Since E and H are related by

$$\begin{aligned} \bar{E}_F &= \eta \bar{H}_F \times \hat{r} = -j\omega\eta(F_\theta \hat{\theta} \times \hat{r} + F_\phi \hat{\phi} \times \hat{r}) \\ &= -j\omega\eta(-F_\theta \hat{\phi} + F_\phi \hat{\theta}) \end{aligned}$$

We have the total field:

$$\bar{E} = -j\omega \left[(A_\theta + \eta F_\phi) \hat{\theta} + (A_\phi - \eta F_\theta) \hat{\phi} \right]$$

Let Js and Ms exist on Sa, and define

$$\bar{P} = \int_{S_a} \bar{E}_a(\bar{r}') e^{j\beta \hat{r} \cdot \bar{r}'} dS', \quad \bar{Q} = \int_{S_a} \bar{H}_a(\bar{r}') e^{j\beta \hat{r} \cdot \bar{r}'} dS'$$

(1) Aperture in ground plane:

$$\begin{aligned} E_\theta &= \frac{j\beta e^{-j\beta r}}{2\pi r} (P_x \cos\phi + P_y \sin\phi) \\ E_\phi &= \frac{j\beta e^{-j\beta r}}{2\pi r} \cos\theta (-P_x \sin\phi + P_y \cos\phi) \end{aligned}$$

(2) Aperture in free-space:

$$\begin{aligned} E_\theta &= \frac{j\beta e^{-j\beta r}}{4\pi r} \frac{1 + \cos\theta}{2} (P_x \cos\phi + P_y \sin\phi) \\ E_\phi &= \frac{j\beta e^{-j\beta r}}{4\pi r} \frac{1 + \cos\theta}{2} (-P_x \sin\phi + P_y \cos\phi) \end{aligned}$$

Example: Uniform rectangular aperture

Aperture field: $\bar{E}_a = \hat{y} E_0, |x| \leq \frac{L_x}{2}, |y| \leq \frac{L_y}{2}$

To calculate P_y :

$$\hat{r} \cdot \bar{r}' = \hat{r} \cdot (\hat{x}x' + \hat{y}y') = ux' + vy'$$

$$\int_{-a}^a e^{j\beta ux'} dx' = a \frac{\sin(\beta ua)}{\beta ua}$$

Hence :

$$\begin{aligned} P_y &= E_0 \int_{-L_x/2}^{L_x/2} e^{j\beta ux'} dx' \times \int_{-L_y/2}^{L_y/2} e^{j\beta vy'} dy' \\ &= E_0 L_x L_y \frac{\sin(\beta u L_x / 2)}{\beta u L_x / 2} \frac{\sin(\beta v L_y / 2)}{\beta v L_y / 2} \end{aligned}$$

The radiation field (for aperture in free-space):

$$E_\theta = \frac{j\beta e^{-j\beta r}}{2\pi r} E_0 L_x L_y \sin\phi \frac{\sin[\beta(L_x/2)u]}{\beta(L_x/2)u} \frac{\sin[\beta(L_y/2)v]}{\beta(L_y/2)v}$$

$$E_\phi = \frac{j\beta e^{-j\beta r}}{2\pi r} E_0 L_x L_y \cos\theta \cos\phi \frac{\sin[\beta(L_x/2)u]}{\beta(L_x/2)u} \frac{\sin[\beta(L_y/2)v]}{\beta(L_y/2)v}$$

E-Plane ($\phi = 90^\circ$): $\cos\phi = 0, E_\phi = 0, u = 0, v = \sin\theta$

$$F_E(\theta) = \frac{\sin[(\beta L_y / 2) \sin\theta]}{(\beta L_y / 2) \sin\theta}$$

H-Plane ($\phi = 0^\circ$): $\sin\phi = 0, E_\theta = 0, u = \sin\theta, v = 0$

$$F_H(\theta) = \cos\theta \frac{\sin[(\beta L_x / 2) \sin\theta]}{(\beta L_x / 2) \sin\theta}$$

Half-Power beamwidth ($L_x, L_y \gg \lambda$):

Since $\sin(1.39)/1.39 = 0.707$,

From $(\beta L_x / 2) \sin \theta_H = 1.39$, $\theta_H = \sin^{-1}(2 \times 1.39 \lambda / 2\pi L_x) \approx 0.443 \lambda / L_x$

$$HP_E = 0.886 \frac{\lambda}{L_y} \text{ (Rad)}, \quad HP_H = 0.886 \frac{\lambda}{L_x} \text{ (Rad)}$$

Note: the HPs are inversely proportional to the size in the corresponding cutting plane.

The approximate HP expressions give good estimations even when L_x, L_y as small as 1.5 wavelength.

$$\text{Directivity: } D_u = \frac{4\pi}{\lambda^2} L_x L_y. \quad (100\% \text{ aperture efficiency})$$

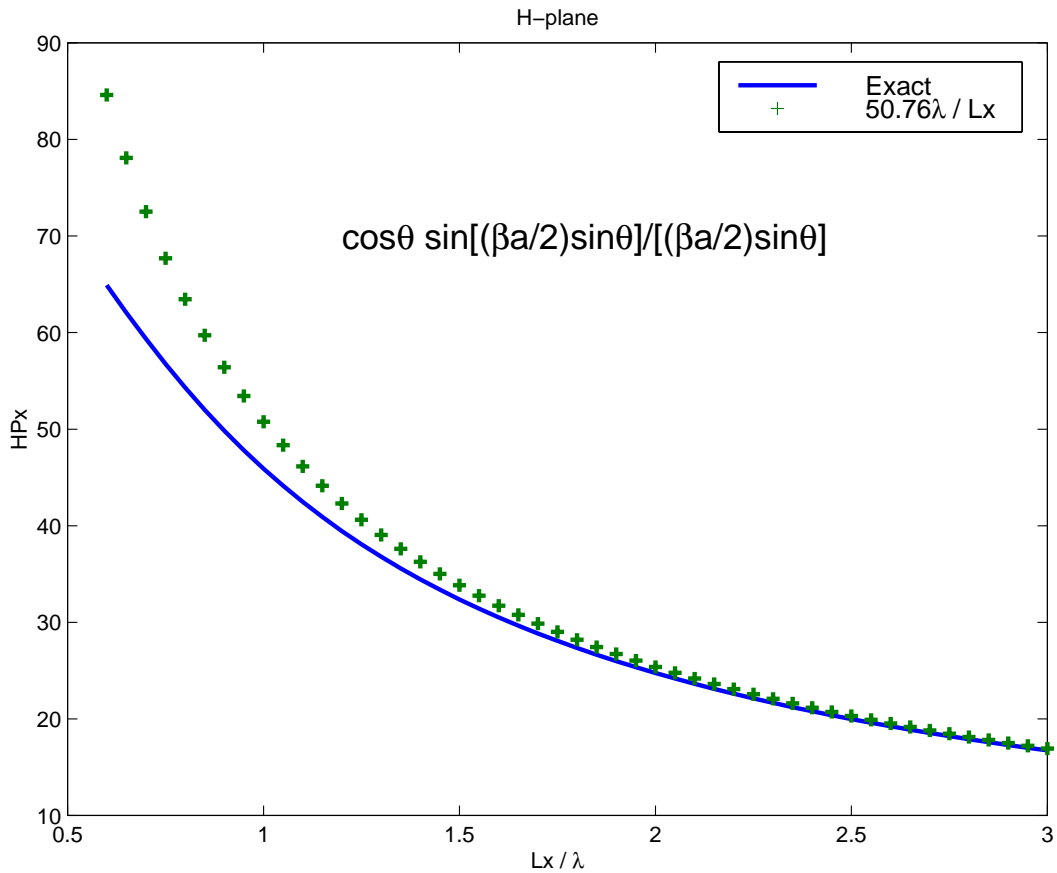
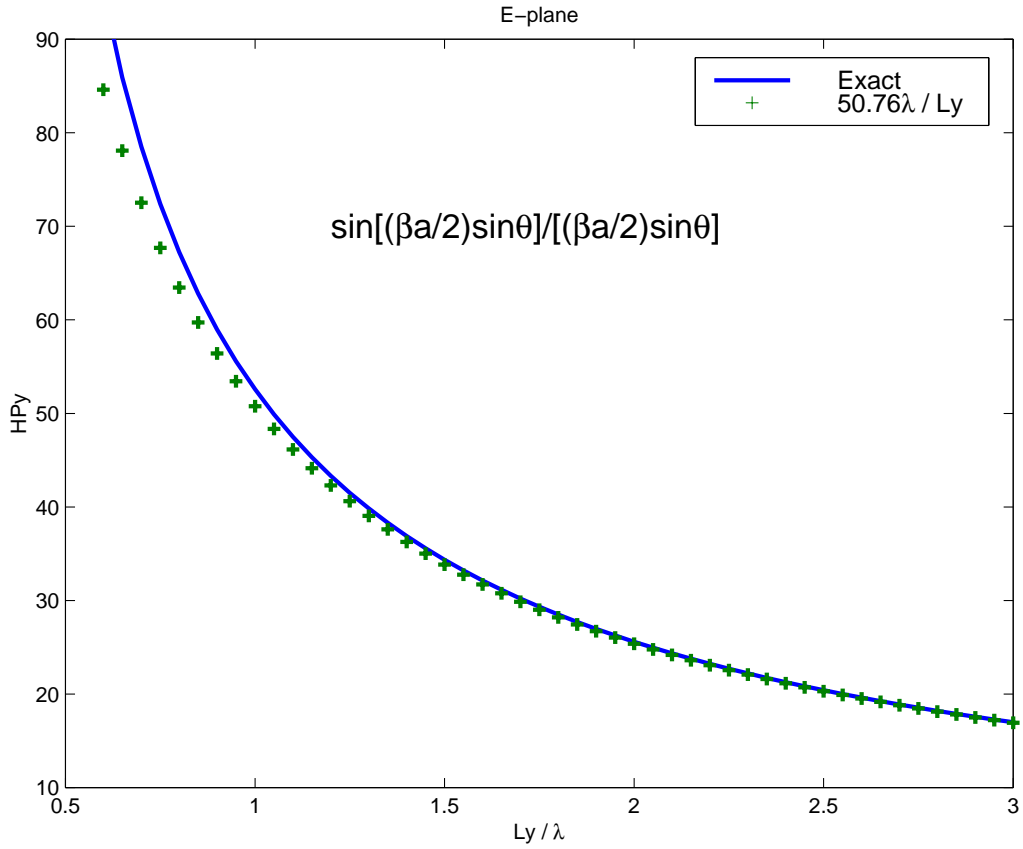
Example: Determine the half-power beam width for E and H plane and the directivity of a rectangular aperture ($10\lambda \times 20\lambda$) with uniform aperture field.

Solution:

$$HP_E = 0.886 \frac{\lambda}{20\lambda} = 0.0443 \text{ (Rad)} = 2.48^\circ,$$

$$HP_H = 0.886 \frac{\lambda}{10\lambda} = 0.0886 \text{ (Rad)} = 4.96^\circ.$$

$$D_u = \frac{4\pi}{\lambda^2} L_x L_y = \frac{4\pi}{\lambda^2} 10\lambda \times 20\lambda = 2513.27 \quad (34\text{dB})$$



Rectangular aperture with tapered aperture field
 Consider constant direction of E:

$$\bar{E}_a = \hat{e} E_x(x) E_y(y), \quad (x, y) \in S_a$$

Then the aperture integration is

$$P_e = \int_{-L_x/2}^{L_x/2} E_x(x') e^{j\beta u x'} dx' \times \int_{-L_y/2}^{L_y/2} E_y(y') e^{j\beta v y'} dy'$$

Example: open-ended rectangular waveguide:
 (aperture on ground plane):

The aperture field is constant in y-direction and cosine variation in x-direction.

$$\begin{aligned} \int_{-b/2}^{b/2} e^{j\beta v y'} dy' &= b \frac{\sin[(\beta b / 2)v]}{(\beta b / 2)v} \\ \int_{-a/2}^{a/2} \cos\left(\frac{\pi x'}{a}\right) e^{j\beta u x'} dx' &= \frac{1}{2} \int_{-a/2}^{a/2} \left[e^{j(\beta u + \pi/a)x'} + e^{j(\beta u - \pi/a)x'} \right] dx' \\ &= \frac{1}{2} \frac{2j \sin[(\beta u a + \pi)/2]}{j(\beta u a + \pi)/2} + \frac{1}{2} \frac{2j \sin[(\beta u a - \pi)/2]}{j(\beta u a - \pi)/2} \\ &= \frac{\cos(\beta a \sin \theta \cos \theta / 2)}{1 - [(\pi / 2)(\beta a \sin \theta \cos \theta / 2)]^2} \end{aligned}$$

For E-plane ($\phi = 90^\circ$):
$$F_E(\theta) = \frac{\sin(\beta b \sin \theta / 2)}{\beta b \sin \theta / 2}$$

For H-plane ($\phi = 0^\circ$):
$$F_H(\theta) = \cos \theta \frac{\sin(\beta a \sin \theta / 2)}{1 - [(\beta a \sin \theta) / \pi]^2}$$

Directivity calculation

- Assumption: (1) Pattern peak near broadside
(2) Aperture relatively large to λ .
(3) Aperture field close to plane wave.

$$D = \frac{4\pi}{\lambda^2} \frac{\left| \int_{S_a} \bar{E}_a dS' \right|^2}{\int_{S_a} |\bar{E}_a| dS'}$$

Example: Using the above formula to calculate the directivity for the open-ended waveguide.

$$\begin{aligned} \left| \int_{-a/2}^{a/2} \cos\left(\frac{\pi x'}{a}\right) dx' \times \int_{-b/2}^{b/2} dy' \right|^2 &= \left(\frac{2a}{\pi} \right)^2 b^2 \\ \int_{-a/2}^{a/2} \cos^2\left(\frac{\pi x'}{a}\right) dx' \times \int_{-b/2}^{b/2} dy' &= \frac{ab}{2} \\ D &= \frac{4\pi}{\lambda^2} \frac{(2a/\pi)^2 b^2}{ab/2} = \frac{4\pi}{\lambda^2} \left(\frac{8ab}{\pi^2} \right) = \frac{4\pi}{\lambda^2} (0.81)ab \end{aligned}$$

In the above, the directivity is reduced by a factor of 0.81 compared to uniform aperture of the same size. This reduction is caused by the tapered aperture, hence the 0.81 here is called aperture taper efficiency that is defined by

$$\epsilon_t = D_t / D_u$$

where

D_t is the directivity of the tapered aperture field.

D_u is the directivity of a uniform field aperture of the same size.

Example: calculate the aperture taper efficiency for a rectangular aperture whose aperture is given by

$$\bar{E}_a = \hat{y} \left(1 - \frac{|x'|}{a/2} \right), \quad (x', y') \in S_a$$

Solution: the x-direction tapering is called triangular tapering.

$$\int_{-a/2}^{a/2} \left(1 - \frac{|x'|}{a/2} \right) dx' = \frac{a}{2}, \quad \int_{-a/2}^{a/2} \left(1 - \frac{|x'|}{a/2} \right)^2 dx' = \frac{a}{3}$$

$$\text{Aperture taper efficiency } \epsilon_t = \frac{1}{ab} \frac{(ab/2)^2}{ab/3} = \frac{3}{4}$$

Example: An aperture operating at 10GHz has physical aperture area of $0.785 m^2$, a gain of 38dB, a directivity of 39dB. Compute (1) Effective aperture, (2) Maximum effective aperture, (3) aperture efficiency, (4) radiation efficiency, (5) aperture taper efficiency.

Solution: At 10GHz, $\lambda = 3 \times 10^8 / (10 \times 10^9) = 0.03m$

$$G = 38dB \Rightarrow G = 6309.6$$

$$D = 39dB \Rightarrow D = 7943.3$$

$$D_u = 4\pi A_p / \lambda^2 = 4\pi(0.785) / (0.03^2) = 10961 \quad (40.4dB)$$

(1) Effective aperture

$$A_e = G\lambda^2 / (4\pi) = 6309.6 \times 0.03^2 / (4\pi) = 0.452m^2$$

(2) Maximum effective aperture:

$$A_{em} = D\lambda^2 / (4\pi) = 7943.3 \times 0.03^2 / (4\pi) = 0.569m^2$$

(3) Aperture efficiency: $\epsilon_{ap} = A_e / A_p = 0.452 / 0.785 = 0.576$

(4) Radiation efficiency: $e_r = G / D = 6309.6 / 7943.4 = 0.794$

(5) Aperture taper efficiency: $\epsilon_t = D / D_u = 7943.4 / 10961 = 0.725$.

H-plane sectoral Horn Antenna

Aperture field distribution:

$$\bar{E}_a = \hat{y}E_0 \cos\left(\frac{\pi x}{A}\right) e^{-j(\beta/2R_1)x^2}$$

$$P_y = E_0 \int_{-A/2}^{A/2} \cos(\pi x'/A) e^{j\beta u x' - j(\beta/2R_1)(x')^2} dx' \times \int_{-b/2}^{b/2} e^{j\beta v y'} dy'$$

Steps to evaluate the integral

$$I = \int_{-A/2}^{A/2} \cos(\pi x'/A) e^{j\beta u x' - j(\beta/2R_1)(x')^2} dx'$$

(1) Convert cosine function into exponentials:

$$\cos(\pi x'/A) = (1/2) \left[e^{j\pi x'/A} + e^{-j\pi x'/A} \right]$$

(2) Write exponential into $p(x'+q)^2 + c$

$$\begin{aligned} (\pi/A + \beta u)x' + (\beta/2R_1)(x')^2 &= (\beta/2R_1) \left[(x')^2 + \frac{\pi/A + \beta u}{\beta/(2R_1)} x' \right] \\ &= (\beta/2R_1) \left[(x')^2 + \frac{1}{2} \frac{2\pi/A + 2\beta u}{\beta/(2R_1)} x' + \left(\frac{2\pi/A + 2\beta u}{\beta/(2R_1)} \right)^2 - \left(\frac{2\pi/A + 2\beta u}{\beta/(2R_1)} \right)^2 \right] \\ &= (\beta/2R_1) \left[x' + \frac{2\pi/A + 2\beta u}{\beta/(2R_1)} \right]^2 - (\beta/2R_1) \left(\frac{2\pi/A + 2\beta u}{\beta/(2R_1)} \right)^2 \\ &= p(x'+q)^2 + c \end{aligned}$$

Where

$$p = (\beta/2R_1), \quad q = \frac{2\pi/A + 2\beta u}{\beta/(2R_1)}, \quad c = -(\beta/2R_1) \left(\frac{2\pi/A + 2\beta u}{\beta/(2R_1)} \right)^2$$

(3) Evaluate the integral into Fresnel integrals:

$$\text{Let } p(x'+q)^2 = (\pi/2)\tau^2$$

$$\begin{aligned} \int_{-A/2}^{A/2} e^{jp(x'+q)^2} dx' &= \sqrt{\frac{\pi}{2p}} \int_{t_1}^{t_2} e^{j(\pi/2)\tau^2} d\tau \\ &= \sqrt{\frac{\pi}{2p}} [C(t_2) - C(t_1) + jS(t_2) - jS(t_1)] \end{aligned}$$

Where

$$t_1 = \sqrt{\frac{\pi}{2p}}(q - A/2), \quad t_2 = \sqrt{\frac{\pi}{2p}}(q + A/2)$$

And the Fresnel integrals are defined by:

$$C(x) = \int_0^x \cos(\pi\tau^2/2) d\tau, \quad S(x) = \int_0^x \sin(\pi\tau^2/2) d\tau$$

The total radiation field is

$$\begin{aligned} E_\theta &= j\beta \frac{e^{-j\beta r}}{4\pi r} (1 + \cos\theta) \sin\phi P_y \\ E_\phi &= j\beta \frac{e^{-j\beta r}}{4\pi r} (1 + \cos\theta) \cos\phi P_y \end{aligned}$$

Directivity: For a given axial length R_1 , there is an optimum value of $A/\text{Wavelength}$ for which the directivity is maximum. The optimum H-plane horn: $A = \sqrt{3\lambda R_1}$.

Explanation: For given R_1 , when A increases from beginning from $A=a$, the aperture size increases, hence the directivity will increase. However when A is too large, the phase error will be so large as to cancel the radiation in the main beam direction, resulting in reduction in directivity.

Computer assignment

Write a computer program to calculate the radiation pattern of H-sectoral horn antenna (H-plane, use equation 7-119).

The input parameters to your program are: a , b , A , λ .

And the output is the radiation pattern as a function of θ from 0 to 80 degrees with increment of 1 degree. Your program may have the following structures:

```
Start of program
Input parameters
Calculate derived parameters (such as  $R_1$ ,  $\beta$ , etc.).
Determine the number of integration points N and dx.
Open a file for output.
Loop for theta=0 to 80
    Perform the numerical integration to get  $f(\theta)$ :
        Sum=0
        Loop over integration grids  $j=1,2,\dots,N$ 
            Calculate  $x'$ 
            Calculate integrand  $h$ 
            Sum=sum +  $W * h$ 
        End loop  $j$ .
        F=abs(sum)
    Write (theta, F) to output file.
End loop theta.
Close file.
End of program.
```

Usually 0.05λ as the initial grid size. Hence the number of grid point is $N = \text{int}[0.5 + A/(0.05\lambda)]$, and the actual grid size is determined by $\Delta x = A/(N - 1)$. A simple method to evaluate the numerical integration is to use the trapezoidal rule. The following FORTRAN code shows the application of the trapezoidal rule to calculate

$$F(\theta) = \int_{-A/2}^{A/2} e^{j\beta x' \sin\theta} dx'$$

```

IMPLICIT NONE
INTEGER I, J, N
REAL WL, A, DX, W, THETA, BETA, PI
COMPLEX F, CX
PARAMETER(PI=3.1415926)
WRITE(*,*) 'Input A, WL=?'
READ(*,*) A,WL
OPEN(1,FILE='sample.ptn',status='unknown')
BETA = 2.0 * PI/WL
N = 0.5 + A/ (0.05 * WL)
IF (N < 2 ) N=2
DX = A / FLOAT(N-1)
DO I=0, 80
  THETA = FLOAT(I)*PI/180.0
  F=0.0
  DO J=1,N
    X= - 05.* A + DX * FLOAT(J-1)
    W=DX
    IF((J .EQ.1).OR.(J.EQ.N) ) W=0.5 * DX
    CX=(0.0,1.0) * BETA * X * SIN(THETA)
    F=F+W * EXP(CX)
  END DO
  WRITE(1,*) THETA, ABS(F)
END DO
CLOSE(1)
STOP
END

```

Circular apertures:

Aperture is on x-y plane (z=0).

Aperture radius=a.

Will consider: uniform and tapered distribution.

1. Uniform aperture field

$$\bar{E}_a = \hat{x}E_0, \quad (x, y) \in S_a$$

$$\bar{P} = \hat{x}E_0 \int_0^a \int_0^{2\pi} e^{j\beta \hat{r} \cdot \bar{r}'} \rho' d\rho' d\phi'$$

Again,

$$\hat{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$$

$$\bar{r}' = \hat{x} \rho' \cos \phi' + \hat{y} \rho' \sin \phi'$$

$$\begin{aligned} \hat{r} \cdot \bar{r}' &= \rho' \sin \theta \cos \phi \cos \phi' + \rho' \sin \theta \sin \phi \sin \phi' \\ &= \rho' \sin \theta \cos(\phi - \phi') \end{aligned}$$

$$\bar{P} = \hat{x}E_0 \int_0^a \left[\int_0^{2\pi} e^{j\beta \rho' \sin \theta \cos(\phi - \phi')} d\phi' \right] \rho' d\rho'$$

$$= \hat{x}E_0 2\pi \int_0^a J_0(\beta \rho' \sin \theta) \rho' d\rho'$$

$$= \hat{x}E_0 (\pi a^2) 2J_1(\beta a \sin \theta) / (\beta a \sin \theta)$$

The radiation field:

$$\bar{E} = (\hat{\theta} \cos \phi - \hat{\phi} \cos \theta \sin \phi) \frac{j\beta e^{-j\beta r}}{4\pi r} E_0 (\pi a^2) \frac{2J_1(\beta a \sin \theta)}{\beta a \sin \theta}$$

$$\text{E-plane pattern } (\phi = 0): \quad F_\theta(\theta) = \frac{2J_1(\beta a \sin \theta)}{\beta a \sin \theta}$$

$$\text{H-plane pattern } (\phi = 90): \quad F_\phi(\theta) = \cos \theta \frac{2J_1(\beta a \sin \theta)}{\beta a \sin \theta}$$

For large apertures, the HP values for E and H plane are almost the same, they equal to

$$HP = 1.02\lambda / (2a) \text{ (rad)}$$

Side lobe level ($a \gg \lambda$): -17.6dB

Directivity: $D_u = \frac{4\pi}{\lambda^2} (\pi a^2)$.

Example: An aperture antenna with uniform aperture field has HP=2 degrees. Find the size and the directivity of the antenna.

Solution: From $HP = 1.02\lambda / (2\alpha)$, $a = 0.61\lambda / HP = 17.48\lambda$

$$D_u = 4\pi(\pi a^2) / \lambda^2 = 4\pi \times \pi \times 17.48^2 = 12061, \text{ (40.8dB)}$$

Tapered aperture field:

Aperture taper efficiency will reduce

SLL will reduce

HP will increase.

Parabolic taper:

$$E_a = \left[1 - \left(\frac{\rho'}{a} \right)^2 \right], \quad f(\theta) = \frac{8J_2(\beta a \sin \theta)}{(\beta a \sin \theta)^2}, \quad \epsilon_t = 0.75, \quad SLL = -24.6dB, \quad HP = 1.27 \frac{\lambda}{2a}$$

Parabolic squared taper:

$$E_a = \left[1 - \left(\frac{\rho'}{a} \right)^2 \right]^2, \quad f(\theta) = \frac{48J_3(\beta a \sin \theta)}{(\beta a \sin \theta)^3}, \quad \epsilon_t = 0.55, \quad SLL = -30.6dB, \quad HP = 1.47 \frac{\lambda}{2a}$$

Parabolic reflector antennas

Parameters: D---Aperture diameter
F/D---Focal length-to-diameter ratio

Points on surface can be specified in two ways:

(1) *by* (ρ', z_f) : $(\rho')^2 = 4F(F - z_f)$

(2) *by* (r_f, θ_f) : $r_f = 2F / (1 + \cos \theta_f) = F \sec^2(\theta_f / 2)$

Relation of ρ' and r_f :

$$\rho' = r_f \sin \theta_f = \frac{2F \sin \theta_f}{1 + \cos \theta_f} = 2F \tan \frac{\theta_f}{2}$$

Normal direction: $\hat{n} = -\bar{r}_f \cos(\theta_f / 2) + \hat{\theta}_f \sin(\theta_f / 2)$

Properties:

- (1) Rays from the focal point "O" will be reflected by the parabolic surface
- (2) All reflected rays are parallel to the axis z_f .
- (3) The ray path from "O" to reflecting points to focal plane have the same distance (that is equal to $2F$).

Feeder pattern: General

$$\bar{F}(\theta_f, \phi_f) = \hat{\theta}_f F_\theta(\theta_f, \phi_f) + \hat{\phi}_f F_\phi(\theta_f, \phi_f)$$

Feeder pattern: Linear:

$$\bar{F}(\theta_f, \phi_f) = \hat{u}_i F_f(\theta_f, \phi_f)$$

Methods to obtain the radiation pattern of the reflector antenna:

- (1) GO/Aperture distribution method
Find the aperture field distribution and integrate over the aperture (flat, focal plane) to get the radiation field.
- (2) PO/Surface current method
Use PO approximation to calculate the current distribution over the reflecting surface (curved) and integrate over the parabolic surface to get the radiation field.

PO/Aperture Distribution Method

Ray tube: formed by rays originated from "O" in a small angular region " $d\Omega$ ". This ray tube has a footprint on the focal plane of " dA ".

Conservation of power: Powers flowing through any cross section in the ray tube are the same.

Using the power conservation concept, we can show that the electric field magnitude over the focal plane is inversely proportional to the distance from "O" to the reflecting surface:

$$E_a(\theta_f) \propto \frac{1}{r_f}$$

The reducing in field magnitude given in the above equation is called spherical spreading loss (because the reflected rays are parallel, the power does not spread over the reflecting part of the ray tube). Based on this analysis, the electric field distribution on the focal plane can be shown to be

$$\bar{E}_a(\theta_f, \phi') = V_0 e^{-j2\beta F} \frac{F_f(\theta_f, \phi')}{r_f} \hat{u}_r \quad (\text{for linear feed})$$

Where \hat{u}_r is the same as the reflected electric field and can be found using the Snell's law:

$$\begin{aligned}\bar{E}_r &= 2(\hat{n} \cdot \bar{E}_i)\hat{n} - \bar{E}_i, \\ \hat{u}_r &= 2(\hat{n} \cdot \hat{u}_i)\hat{n} - \hat{u}_i\end{aligned}$$

$$\begin{cases} E_\theta = \frac{j\beta e^{-j\beta r}}{4\pi r} \frac{1+\cos\theta}{2} (P_x \cos\phi + P_y \sin\phi) \\ E_\phi = \frac{j\beta e^{-j\beta r}}{4\pi r} \frac{1+\cos\theta}{2} (-P_x \sin\phi + P_y \cos\phi) \end{cases}$$

$$\begin{aligned} \bar{P} &= \int_{S_a} \bar{E}_a(\bar{r}') e^{j\beta \hat{r} \cdot \bar{r}'} dS' \\ &= V_0 \int_0^{2\pi} \int_0^a \frac{F_f(\theta_f, \phi')}{r_f} \hat{u}_r e^{j\beta \rho' \sin\theta \cos(\phi - \phi')} \rho' d\rho' d\phi' \end{aligned}$$

In the above equation,

θ, ϕ ----- Angle of observation

ρ', ϕ' ----- Integration variables

r_f, θ_f ----- Variables related to ρ', ϕ' :

$$r_f = (4F^2 + \rho'^2)/(4F), \quad \theta_f = 2 \tan(\rho'/2F)$$

Example:

Let the feed be a short dipole the is parallel to \hat{x}_f - axis. Then the primary pattern is given by

$$F_f(\theta_f, \phi') = \sqrt{1 - \cos^2 \theta_f \sin^2 \phi'}$$

$$\hat{u}_i = \frac{\hat{r}_f (\hat{r}_f \cdot \hat{x}_f) - \hat{x}_f}{\sqrt{1 - (\hat{r}_f \cdot \hat{x}_f)^2}}$$

For axial symmetric feed pattern (primary pattern),

$F_f(\theta_f)$ is independent of variable ϕ' , then

$$\bar{P}(\theta, \phi) = 2\pi V_0 \int_0^a \frac{F_f(\theta_f)}{r_f} \hat{u}_r J_0(\beta \rho' \sin\theta) \rho' d\rho'$$