EE461g: Introduction to Electronic Circuits

Lecture 1: Circuit Analysis Review

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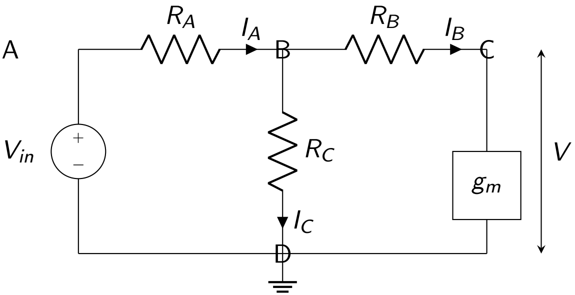
Outline

Example Circuit with Square-Law Device

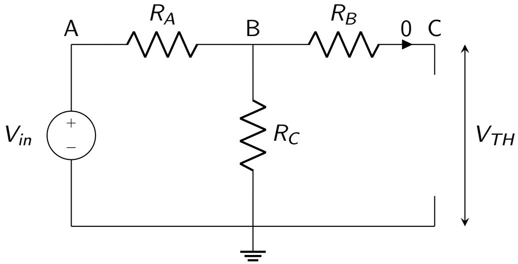


Square-Law Device:

$$I = \begin{cases} g_m(V - V_t)^2 & \text{if } V > V_t \\ 0 & \text{otherwise} \end{cases}$$

Thevenin Equivalent: Finding V_{TH}

Step 1: Remove the square-law device (open circuit at C)



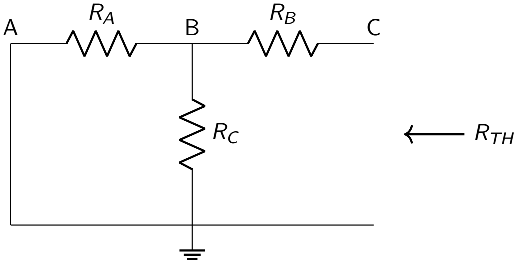
Analysis:

- No current through R_B (open circuit at C)
- Voltage divider: $V_B = V_C = V_{in} \cdot \dfrac{R_C}{R_A + R_C}$

$$\boxed{V_{TH} = V_{in} \cdot \frac{R_C}{R_A + R_C}}$$

Thevenin Equivalent: Finding R_{TH}

Step 2: Short the voltage source, find resistance looking into C-D

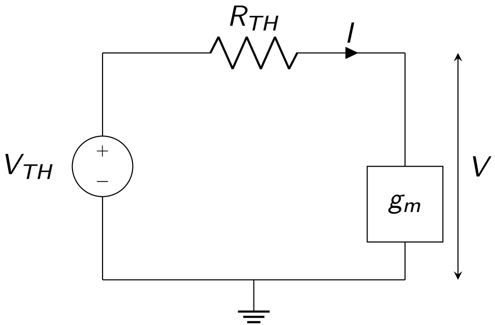


Analysis:

- From C, looking back: R_B in series with $(R_A \| R_C)$

$$\boxed{R_{TH} = R_B + \frac{R_A \cdot R_C}{R_A + R_C} = R_B + (R_A \| R_C)}$$

Thevenin Equivalent Circuit



Where:

$$V_{TH} = V_{in} \cdot \frac{R_C}{R_A + R_C}$$

$$R_{TH} = R_B + (R_A \| R_C)$$

Analyzing Non-Linear Circuits

Problem: The square-law device has two possible states:

$$I = \begin{cases} g_m(V - V_t)^2 & \text{if } V > V_t \quad \text{(ON)} \\ 0 & \text{if } V \leq V_t \quad \text{(OFF)} \end{cases}$$

Approach: Proof by Contradiction

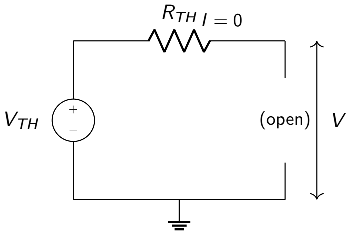
1. **Guess** which state the device is in
2. **Solve** the circuit assuming that guess is correct
3. **Test** if the solution is consistent with our guess
4. If **contradiction** found $\rightarrow$ go back and try the other state

Let's sweep V_{in} from $-\infty$ to $+\infty$ and find V and I.

Guess #1: Device is OFF ($I = 0$)

Assumption: $V \leq V_t$, so $I = 0$

Replace device with open circuit:



Analysis with $I = 0$:

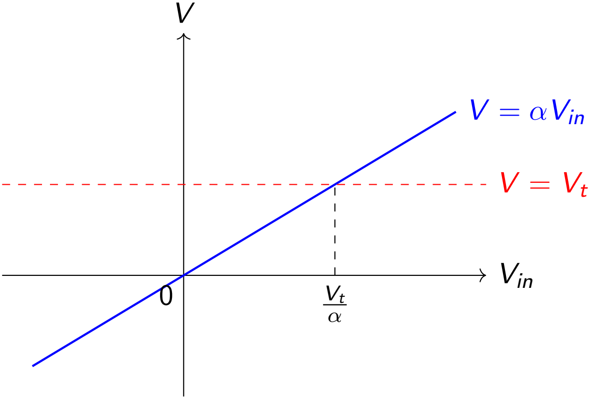
- No current through $R_{TH} \Rightarrow$ no voltage drop across R_{TH}
- Therefore: $V = V_{TH}$

$$\boxed{V = V_{TH} = V_{in} \cdot \frac{R_C}{R_A + R_C}}$$

Guess #1: Plot V vs V_{in}

With the device OFF, we have $V = V_{TH} = \alpha \cdot V_{in}$

where $\alpha = \dfrac{R_C}{R_A + R_C}$ (note: $0 < \alpha < 1$)



Question: Is our guess ($V \leq V_t$) always valid?

Test Guess #1: Check for Contradiction

Our guess assumed: $V \leq V_t$
Our solution gave: $V = \alpha \cdot V_{in}$

Test: When is $V > V_t$? (contradiction!)

$$\alpha \cdot V_{in} > V_t$$

$$V_{in} > \frac{V_t}{\alpha}$$

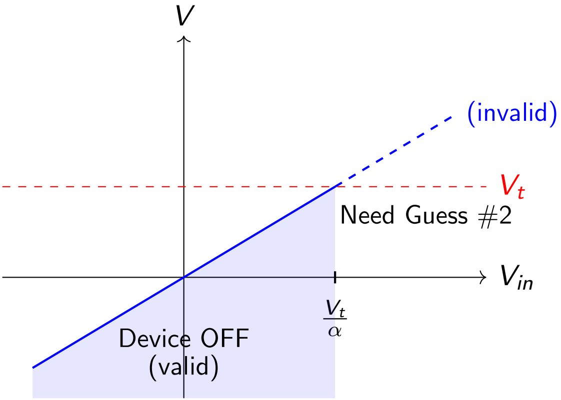
Contradiction Found!

For $V_{in} > \dfrac{V_t}{\alpha}$, our solution gives $V > V_t$,
but we assumed $V \leq V_t$ (device OFF).

$\Rightarrow$ Our guess is **wrong** in this region!

Conclusion: Guess #1 is only valid for $V_{in} \leq \dfrac{V_t}{\alpha}$

Guess #1: Valid Region



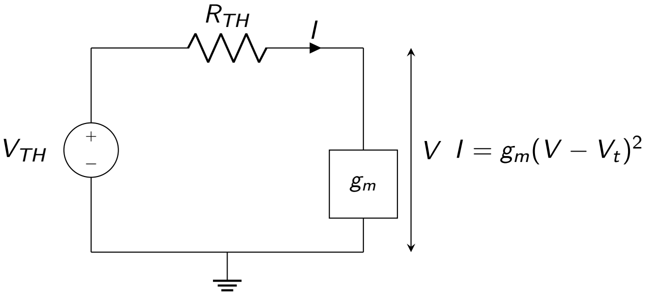
For $V_{in} \leq \frac{V_t}{\alpha}$**:** Device is OFF

$$V = \alpha \cdot V_{in}, \quad I = 0 \quad \checkmark$$

Guess #2: Device is ON ($V > V_t$)

For $V_{in} > \dfrac{V_t}{\alpha}$**:** Try the other state

Assumption: $V > V_t$, so $I = g_m(V - V_t)^2$



Apply KVL:

$$V_{TH} = I \cdot R_{TH} + V = g_m R_{TH}(V - V_t)^2 + V$$

Guess #2: Solve for V

KVL equation: $V_{TH} = g_m R_{TH}(V - V_t)^2 + V$

Let $x = V - V_t$:

$$V_{TH} = g_m R_{TH} \cdot x^2 + (x + V_t)$$

$$g_m R_{TH} \cdot x^2 + x + (V_t - V_{TH}) = 0$$

Quadratic formula:

$$x = \frac{-1 + \sqrt{1 + 4 g_m R_{TH}(V_{TH} - V_t)}}{2 g_m R_{TH}}$$

Therefore:

$$\boxed{V = V_t + \frac{-1 + \sqrt{1 + 4 g_m R_{TH}(V_{TH} - V_t)}}{2 g_m R_{TH}}}$$

Current: $I = g_m(V - V_t)^2 = g_m \cdot x^2$

Test Guess #2: Verify Consistency

Our guess assumed: $V > V_t$ (i.e., $x > 0$)

Check: Is $x > 0$ when $V_{in} > \frac{V_t}{\alpha}$?

$$x = \frac{-1 + \sqrt{1 + 4g_m R_{TH}(V_{TH} - V_t)}}{2g_m R_{TH}}$$

For $x > 0$, we need:

$$\sqrt{1 + 4g_m R_{TH}(V_{TH} - V_t)} > 1$$

$$V_{TH} > V_t \quad \Rightarrow \quad V_{in} > \frac{V_t}{\alpha} \quad \checkmark$$

No Contradiction!

When $V_{in} > \frac{V_t}{\alpha}$, we get $x > 0$, which means $V > V_t$.
This is consistent with our guess. $\checkmark$

Complete Solution: V and I vs V_{in}

Define: $\alpha = \dfrac{R_C}{R_A + R_C}$, $\quad R_{TH} = R_B + (R_A \| R_C)$

Region 1: $V_{in} \leq \dfrac{V_t}{\alpha}$ (Device OFF)

$$V = \alpha \cdot V_{in}, \qquad I = 0$$

Region 2: $V_{in} > \dfrac{V_t}{\alpha}$ (Device ON)

$$V = V_t + \frac{-1 + \sqrt{1 + 4g_m R_{TH}(\alpha V_{in} - V_t)}}{2g_m R_{TH}}$$

$$I = g_m (V - V_t)^2$$

Simulation Parameters

Circuit Component Values:

Parameter	Value	Description
R_A	$100\ \Omega$	Input resistor
R_B	$100\ \Omega$	Series resistor to device
R_C	$1000\ \Omega$	Shunt resistor
g_m	$10\ \text{mA/V}^2$	Transconductance parameter
V_t	$1.0\ \text{V}$	Threshold voltage

Derived Thevenin Parameters:

$$\alpha = \frac{R_C}{R_A + R_C} = \frac{1000}{100 + 1000} = 0.909$$

$$R_{TH} = R_B + (R_A \| R_C) = 100 + \frac{100 \times 1000}{1100} = 190.9\ \Omega$$

$$V_{in,threshold} = \frac{V_t}{\alpha} = \frac{1.0}{0.909} = 1.1\ \text{V}$$

SPICE Circuit Netlist

```
* EE461g Lecture 1 - Square Law Device Circuit

* Input voltage source
Vin A 0 DC 5

* Resistor network
Ra A B 100
Rb B C 100
Rc B 0 1000

* Square-law device: I = gm*(V-Vt)^2 for V > Vt
* Using behavioral current source (gm = 10mA/V^2)
Bsqlaw C 0 I = 10m * uramp(V(C)-1) * uramp(V(C)-1)

* DC sweep from -2V to 10V
.dc Vin -2 10 0.01

* Print/plot outputs
.print dc V(A) V(B) V(C) I(Bsqlaw)
.plot dc V(C)
.plot dc I(Bsqlaw)

.end
```

Thevenin Parameter Verification

KVL Check: $V_{TH} = I \cdot R_{TH} + V$

V_{in} (V)	State	V_{TH} (V)	V (V)	I (mA)	$I \cdot R_{TH} + V$
0.0	OFF	0.000	0.000	0.00	0.000 ✓
1.0	OFF	0.909	0.909	0.00	0.909 ✓
1.1	OFF	1.000	1.000	0.00	1.000 ✓
2.0	ON	1.818	1.443	1.96	1.818 ✓
5.0	ON	4.545	2.126	12.67	4.545 ✓
10.0	ON	9.091	2.813	32.88	9.091 ✓

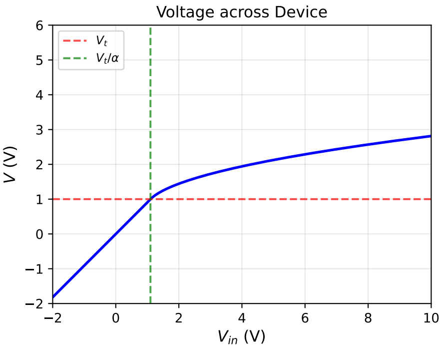
Direct Circuit vs Thevenin Comparison:

V_{in}	Direct (Nodal)		Thevenin	
(V)	V (V)	I (mA)	V (V)	I (mA)
2.0	1.443195	1.9642	1.443195	1.9642
5.0	2.125804	12.6744	2.125804	12.6744
10.0	2.813352	32.8824	2.813352	32.8824

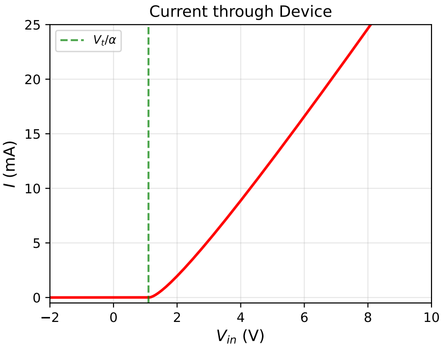
Result: Thevenin equivalent verified — errors $< 10^{-12}$ V

Graphical Summary (SPICE Simulation)

Voltage V vs V_{in}:



Current I vs V_{in}:



Key Insight:

- Below threshold ($V_{in} < V_t/\alpha$): V follows V_{in} linearly, $I = 0$
- Above threshold: V increases slowly (current "loads" the circuit)

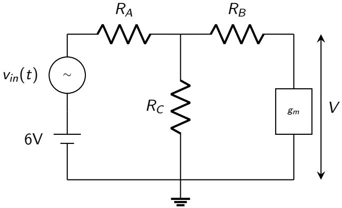
Summary: Proof by Contradiction Method

1. **Identify** the possible states of the non-linear device
2. **Guess** a state and replace device with equivalent
   - OFF: replace with open circuit
   - ON: use the device equation
3. **Solve** the circuit with your guess
4. **Test** if solution satisfies the guess condition
   - If yes $\rightarrow$ solution is valid in that region
   - If no $\rightarrow$ contradiction! Try other state
5. **Combine** solutions from all regions

This method works for **any** piecewise non-linear device!

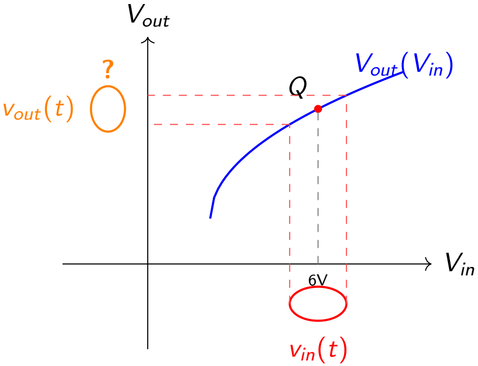
Goal: Linearize the non-linear device around an operating point

Input: **DC + small AC signal**



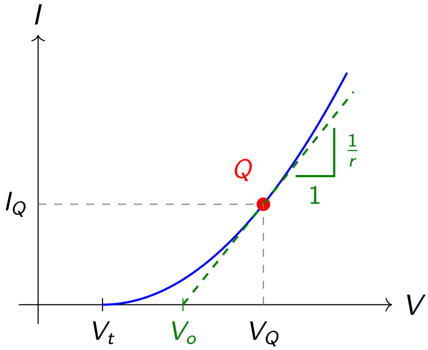
$$V_{in}(t) = 6\text{V} + v_{in}(t)$$

Graphical interpretation:



Question: What is the amplitude of $v_{out}(t)$?

Small Signal Model: Device Linearization



Square-law device:

$$I = g_m(V - V_t)^2$$

Linear approximation:

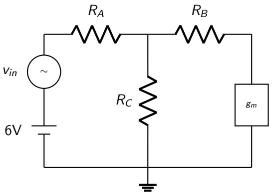
$$I = \frac{V - V_o}{r}$$

Small signal resistance:

$$r = \left.\frac{1}{dI/dV}\right|_Q = \frac{1}{2g_m(V_Q - V_t)}$$

Key idea: Near Q, the tangent line approximates the curve for small signals

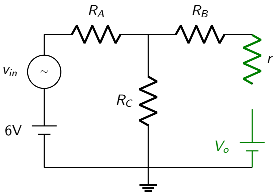
Original circuit:



Non-linear device g_m

$\Rightarrow$

With linear model:



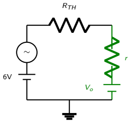
Linear model: $V_o + r$

Result: The non-linear device is replaced by a DC voltage V_o in series with resistor r

Small Signal Model: Superposition

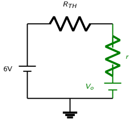
Since the circuit is now linear, we can use superposition!

Full circuit:



$=$

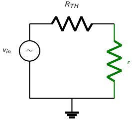
DC only:



(Q-point)

$+$

AC only:



(small signal)

Key Insight

We only need to solve the **AC circuit** — the DC part just gives us the Q-point!

$$v_{out} = v_{in} \cdot \frac{r}{R_{TH} + r}$$

Step 1: Find the DC operating point (Q-point)

From simulation at $V_{in} = 6$ V:

$$V_{TH} = \alpha \cdot V_{in} = 0.909 \times 6 = 5.455 \text{ V}$$

$$V_Q = 2.288 \text{ V}$$

$$I_Q = g_m(V_Q - V_t)^2 = 10 \text{ mA/V}^2 \times (1.288)^2 = 16.59 \text{ mA}$$

Voltage above threshold: $V_Q - V_t = 2.288 - 1.0 = 1.288$ V

Operating Point Summary

$$V_Q = 2.288 \text{ V}, \qquad I_Q = 16.59 \text{ mA}$$

Small Signal Model: Derivation

Device small signal resistance:

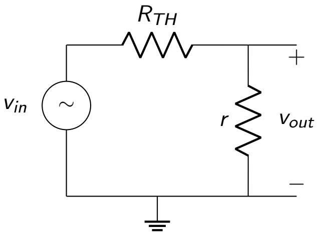
$$r = \left.\frac{1}{dI/dV}\right|_{V=V_Q} = \frac{1}{2g_m(V_Q - V_t)}$$

At our operating point:

$$r = \frac{1}{2 \times 10\ \text{mA/V}^2 \times 1.288\ \text{V}}$$

$$= \boxed{38.82\ \Omega}$$

Small signal circuit:

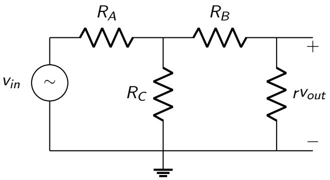


Small Signal Gain (Thevenin circuit)

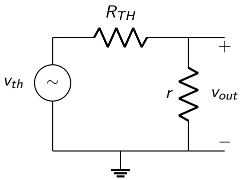
$$\frac{v_{out}}{v_{th}} = \frac{r}{R_{TH} + r} = \frac{38.82}{190.9 + 38.82} = 0.169$$

Small Signal Gain: From Original Input

Original circuit input:



Thevenin equivalent:



where $v_{th} = \alpha \cdot v_{in}$

Two-step gain calculation:

$$\frac{v_{out}}{v_{in}} = \underbrace{\frac{v_{th}}{v_{in}}}_{\alpha} \times \underbrace{\frac{v_{out}}{v_{th}}}_{\frac{r}{R_{TH}+r}} = \alpha \cdot \frac{r}{R_{TH} + r} = 0.909 \times 0.169 = \boxed{0.154}$$

Summary: Small Signal Model

1. **Find the Q-point** (DC operating point)
   - Solve the DC circuit to find V_Q and I_Q
2. **Linearize the device** at the Q-point
   - Compute small signal resistance: $r = \left.\dfrac{1}{dI/dV}\right|_Q = \dfrac{1}{2g_m(V_Q - V_t)}$
3. **Replace device with resistor** r
   - Short all DC sources (voltage $\rightarrow$ short, current $\rightarrow$ open)
4. **Analyze the linear AC circuit**
   - Use voltage divider, superposition, etc.
   - Small signal gain: $\dfrac{v_{out}}{v_{in}} = \alpha \cdot \dfrac{r}{R_{TH} + r}$

Key Result at $V_{in} = 6$ V

$r = 38.82\ \Omega, \qquad \dfrac{v_{out}}{v_{in}} = 0.154$