

Preserving Strong Connectivity in Directed Proximity Graphs

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Abstract—This paper proposes a method to maintain the strong connectivity property of a mobile robot ad hoc network in the presence of disturbances or additional control goals. Each robot has a communication range modeled by an n -dimensional sphere centered at the robot. The spheres for different robots may have different radii, resulting in a directed communication network. The work is based on two concepts. The first is the structure of the Perron vector for reducible stochastic matrices. The second is the design of nonlinear controllers that ensure that two robots remain within a certain distance of each other despite disturbances. The results are supported by analysis and simulations.

I. INTRODUCTION

Multi-robot systems have been proposed as an effective solution to perform various tasks in remote or inhospitable conditions. The scenario proposed involves multiple robots that cooperate to achieve some task.

It is typical to assume that the multiple robots can exchange information through communication. The robots are equipped with wireless communication devices, and hence the robots form a mobile ad-hoc communication network. A simple model for the communication links in such a network is one where an edge exists from one robot to another if the Euclidean distance between the robots is smaller than the communication radius of the former robot. Since the edge weights of the resulting graph depend on the distance between the corresponding nodes, the graph arising from such a mechanism is often called a proximity graph.

Several algorithms that enable teams of multiple robots to complete a task implicitly assume that the communication network formed by the robots is suitably connected. Specifically, any robot in the network should be able to send information to any other robot in the network. This ability to exchange information is often crucial to the cooperation and coordination of the robots [1]. Some distributed optimization and estimation problems also require that the directed communication graphs involved possess suitable connectivity properties [2], [3].

The topology of a proximity graph depends on the positions of the robots. As the robots move, the graph topology will change, and the connectivity properties that various algorithms rely on may be lost. This phenomenon gives rise to the connectivity control problem. In the next two subsections, we briefly outline the related work on connectivity control in directed and undirected graphs, and the contributions towards this topic in this paper.

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A. Related Work

Very little work has been presented to address the issue of maintaining strong connectivity in dynamic directed proximity graphs. In [4], a combinatorial approach is taken, noting that the estimation of minimum spanning subgraphs is NP-hard, and hence suboptimal spanning subgraphs are obtained. No control method is proposed. In [5], the authors aim to artificially balance the graph, and then extend the gradient-based control in [6] to keep the real part of the algebraic connectivity $\lambda_2(\mathcal{L})$ non-zero, thus guaranteeing preservation of strong connectivity. The work assumes that each robot is responsible for maintaining only its out-edges, and not the in-edges. As a result, in certain cases edges can only be preserved by allowing the agents to increase their communication radius.

The problem of maintaining connectivity for undirected networks has received much attention. A rather thorough review can be found in [7]. Early methods preserved the initial edges for all time, with the option of preserving any new edges that are formed. An improvement involved estimating which edges are critical to maintaining connectivity and which are not, and only preserving those critical edges [8]. The set of critical edges is fixed for all time. Later works were interested in making the connectivity independent of the initial graph topology. Such methods proposed controlling the algebraic connectivity of the network instead of the individual edges. The set of edges at some time $t = T$ may be completely different from those existing at the initial time $t = 0$, under such control methods. A significant connectivity increasing method was proposed by [9] and modified in [6], [10], [11] to allow link deletions in a smooth manner. A similar method was proposed in [12] and modified in [13], [14]. Another advantage of these works is that the robots can move continuously, without needing to stop, assess the connectivity and compute a connectivity-preserving motion.

Our goal is to achieve the performance of these latter works for the case of preserving strong connectivity in directed communication networks. We want to allow the graph topology to change arbitrarily whenever this would not lead to loss of strong connectivity. If loss of some edges may lead to loss of strong connectivity, we want to maintain strong connectivity by preserving these edges without preserving a large number of additional edges.

While we focus on edges that depend on Euclidean distances in this paper, our method can be extended to communication models involving more complex functions of the robot positions [15] (see Section V).

B. Contributions

We propose a method of maintaining strong connectivity for the communication network formed by mobile robots. The method involves two contributions. First, we design a control law that can ensure that any two cooperating robots always remain within a certain distance from each other. This guarantee holds even if multiple pairs of robots must remain within a certain distance from each other, and each robot's dynamics is driven by an additional bounded control term or bounded disturbance terms. Second, we show how to use the the Perron vector of an irreducible stochastic graph matrix to decide which inter-robot distances must be preserved.

II. PRELIMINARIES

A. Directed Graphs

A directed graph $G = (V, E)$ is a tuple consisting of a set of vertices V (also called nodes) and a set of edges E (also called links). If there are N nodes, then we can label them using integers from 1 to N , so that V can be identified with the set $\{1, 2, \dots, N\}$. An edge $e \in E$ is an ordered pair (i, j) , which indicates that a connection exists that starts at node i and ends at node j , where $i, j \in V$. More precisely, the edge (i, j) is an out-edge for node i and an in-edge for node j .

When edges in a directed graph have the property that $(i, j) \in E$ implies $(j, i) \in E$, the graph is said to be undirected. In this case, an edge (i, j) can also be viewed as an unordered pair. Undirected graphs can be viewed as special cases of directed graphs. Thus, in this paper, a graph refers to a directed graph unless specified.

We can define two neighbor sets for each node $i \in V$. These are the set of out-neighbors denoted by $\mathcal{N}_i^{\text{out}}$ and the set of in-neighbors denoted by $\mathcal{N}_i^{\text{in}}$. If a directed edge (i, j) exists, then node j is an out-neighbor of node i , and node i is an in-neighbor of node j . Formally,

$$\mathcal{N}_i^{\text{out}} = \{j \in V : (i, j) \in E\}, \text{ and} \quad (1)$$

$$\mathcal{N}_i^{\text{in}} = \{j \in V : (j, i) \in E\}. \quad (2)$$

For an undirected graph, $\mathcal{N}_i^{\text{out}} = \mathcal{N}_i^{\text{in}} = \mathcal{N}_i$.

Definition 1. A *path* is a sequence of edges such that if an edge in the path ends at node i , then the next edge in the sequence starts at node i .

Thus, a path begins on some node i and ends on some other node j (or possibly i), while passing through intermediate nodes as dictated by the edges in the sequence. The paths in the graph give rise to notions of connectivity in a graph. The notion of connectivity differs between directed and undirected graphs.

Definition 2. An undirected graph G is *connected* if a path exists between any two nodes.

Directed networks have multiple definitions of connectivity [16]:

Definition 3. A directed graph G is *weakly connected* if a path exists from any node to any other node when the direction of each edge in the path is disregarded.

For clarity, a path will be called a *directed path* when the directions of the edges must be respected.

Definition 4. A directed graph $G = (V, E)$ is *quasi-strongly connected* if there exists a node $v \in V$ such that a directed path exists from any node $u \in V$ to v .

Definition 5. A directed graph G is *strongly connected* if a directed path exists from any node to any other node.

B. Mobile Robot Networks

Consider a team of N mobile robots. The configuration of the i^{th} robot is given by $q_i \in \mathbb{R}^n$. We can stack the N configurations of the N robots together in an obvious way to obtain the configuration of the team $q \in \mathbb{R}^{Nn}$.

The robots are assumed to have first order dynamics given by

$$\dot{q}_i = u_i^c + u_i^e. \quad (3)$$

where $u_i^c \in \mathbb{R}^n$ is a control term to be designed and $u_i^e \in \mathbb{R}^n$ is a bounded vector representing additional control objectives and/or disturbances.

We wish to model the communication network G formed by this team of mobile robots. The nodes V of the network are the robots, and edges E in the network correspond to the ability of the robots to send information to other robots. Robot i is able to send information to any other robot within a distance of R_i from the center of the robot. Let the distance between the i^{th} and j^{th} robot be $d_{ij} = d_{ji}$. Thus, if $d_{ij} < R_i$, then robot i can send information to robot j . In other words $(i, j) \in E$ if and only if $d_{ij} = \|q_i - q_j\| < R_i$, where $i, j \in V = \{1, 2, \dots, N\}$. A graph obtained in this way is often referred to as a proximity-graph.

The configuration q of the mobile robot team and the parameters R_i , $i \in \{1, 2, \dots, N\}$ determine the directed proximity graph G . The directed graph G is not a fixed graph, but rather is dynamic because it depends on the time-varying state $q(t)$. The dynamic nature of the graph motivates the problem statement defined in the next section.

We make two assumptions about the sensing capabilities of the robots:

- A1 The i^{th} robot can measure the relative location of any robot inside its communication range defined by the sphere of radius R_i .
- A2 If robot j can receive information from robot i , then robot j can estimate the direction towards robot i (in the body frame of robot j).

III. CONTROL PROBLEM

Consider a team of N robots with communication radii R_i , $i \in \{1, 2, \dots, N\}$. Let the dynamics of the i^{th} robot be given by (3). Let the configuration of the team be $q \in \mathbb{R}^{Nn}$ and the corresponding directed graph at any time t be $G(q(t))$. Let t_0 be some time instant, such that $G(q(t_0))$ is strongly connected.

Control problem:

Let there exist some $M \in \mathbb{R}$ where $0 < M < \infty$ such that $\|u_i^e(t)\| < M \forall i \in \{1, 2, \dots, N\}$ and $\forall t \geq t_0$. Design a feedback control $u_i^c(t) \in \mathbb{R}^n$ for the robots such that $G(q(t))$ is strongly connected for all $t \geq t_0$.

IV. ALGEBRAIC CONNECTIVITY

In this section, we show how algebraic graph theory can be used to decide which edges of a strongly connected graph must be preserved in order to preserve strong connectivity.

A. Graph Matrices

Given a directed graph $G = (V, E)$, we can assign a weight w_{ij} to each edge $(j, i) \in E$. Once the edge weights w_{ij} are defined, the adjacency matrix $A_w(G) = \{a_{ij}\} \in \mathbb{R}^{N \times N}$ is given by

$$a_{ij} = \begin{cases} w_{ij} & \text{if } (j, i) \in E, \text{ or} \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Note that the non-zero entries of the i^{th} row correspond to the edges directed towards node i , which are the in-edges of i . It is also possible to define the i^{th} row based on out-edges, however we do not use that formulation.

If for all $i, j \in V$, $w_{ij} = w_{ji}$ then the graph is said to be undirected, and the graph matrices will be symmetric. If there are two nodes such that $w_{ij} \neq w_{ji}$ then the graph is called a directed graph, or digraph for short.

The Laplacian $\mathcal{L}_w(G) \in \mathbb{R}^{N \times N}$ of a directed graph can be derived from the adjacency matrix $A_w(G)$ and is given by

$$\mathcal{L}_w(G) = D_w(G) - A_w(G). \quad (5)$$

where $D_w(G)$ is a diagonal matrix whose i^{th} diagonal element is $\sum_{j=1}^N w_{ij}$. Due to the definition of A_w , the matrix \mathcal{L}_w is often known as the in-Laplacian, however we do not refer to it as such in the paper. The Laplacian $\mathcal{L}_w(G)$ always has an eigenvalue at 0, corresponding to a right eigenvector given by $\frac{1}{\sqrt{N}} \mathbf{1}_N$, where $\mathbf{1}_N \in \mathbb{R}^N$, $\mathbf{1}_N = [1, \dots, 1]^T$.

The remaining eigenvalues of \mathcal{L}_w may be complex, and are ordered based on their absolute value. Let the i^{th} smallest eigenvalue of a matrix A be denoted by $\lambda_i(A)$. The second smallest eigenvalue of $\mathcal{L}_w(G)$ is denoted by $\lambda_2(\mathcal{L}_w)$. In the case of undirected graphs, it is called the Fiedler value of the graph [17].

Note that we will henceforth drop the subscript w from these matrices, since in the rest of the paper we assume that all matrices are constructed using w_{ij} unless specified.

We can create a non-negative row-stochastic [18] matrix $S \in \mathbb{R}^{N \times N}$, by using the transformation

$$S = I_N - \varepsilon \mathcal{L} \quad (6)$$

where I_N is the identity matrix of size N and $\varepsilon > 0$ is a sufficiently small number. This transformation was used in [1], with a view to analysis of discrete-time consensus protocols. It turns out that the same matrix is useful in the estimation of $\lambda_2(\mathcal{L})$ [19]. Assuming that each edge weight w_{ij} is bounded above by 1, then selecting $\varepsilon \leq 1/N$ ensures S is non-negative.

We define the weights of the directed graph by using bump functions. In particular, we choose the weights to be $w_{ij} = \psi_{(0, R_j)}(d_{ij})$, where $\psi : \mathbb{R}_+ \rightarrow [0, 1]$ is given by

$$\psi_{(\rho_1, \rho_2)}(x) = \begin{cases} 1 & \text{if } x \leq \rho_1, \\ \frac{\exp(-\frac{1}{\rho_2-x})}{\exp(-\frac{1}{\rho_2-x}) + \exp(-\frac{1}{\rho_1-x})} & \text{if } \rho_1 < x < \rho_2, \text{ or} \\ 0 & \text{if } \rho_2 \leq x. \end{cases} \quad (7)$$

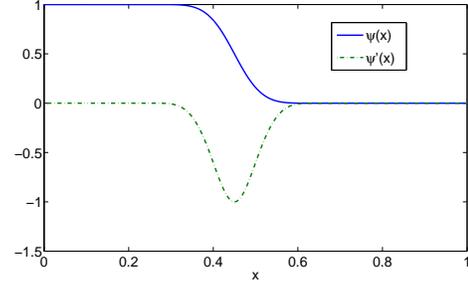


Fig. 1. Bump function for $\rho_1 = 0.2m$ and $\rho_2 = 0.7m$.

One of the advantages of bump functions is that they are smooth objects and can thus be differentiated as many times as required. If we take the distance d_{ij} between two robots as the domain of $\psi_{(\rho_1, \rho_2)}(x)$, we obtain a smooth weighting $w_{ij} = \psi_{\rho_1, \rho_2}(d_{ij})$ from full connectivity to no connectivity for any two robots, as seen in Figure 1.

The choice for ψ above is motivated by the communication model we use. However, the results in the remainder of this section are valid for any edge weights that are positive and bounded. Given such a bound, one can choose ε in (6) to ensure that S is non-negative even for dynamic state-dependent graphs.

B. Algebraic Connectivity for Directed Graphs

For an undirected graph to be connected, at most one eigenvalue of its graph Laplacian \mathcal{L} can be zero. The second smallest eigenvalue $\lambda_2(\mathcal{L})$ thus becomes a measure of connectivity in the graph. Note that for an undirected graph, \mathcal{L} is symmetric so that all its eigenvalues are real.

In the case of directed graphs, the matrices A and \mathcal{L} need not be symmetric, and $\lambda_2(\mathcal{L})$ may be complex. The magnitude of $\lambda_2(\mathcal{L})$ does not give us any information about whether the graph is strongly connected or not. However, it does indicate whether the network is quasi-strongly connected (and therefore weakly connected) or not. This property is due to the following theorem derived from results in [20]:

Theorem IV.1 ([20]). *Consider a directed graph G with Laplacian \mathcal{L} as defined in (5). The graph G is quasi-strongly connected if and only if $|\lambda_2(\mathcal{L})| > 0$*

For undirected networks, $\lambda_2(\mathcal{L})$ can be used to determine connectivity. Techniques to estimate $\lambda_2(\mathcal{L})$ in a decentralized manner [9] and use this estimate for connectivity control [6], [10], [11], [14] have been proposed, as mentioned earlier. For balanced networks (sum of rows and columns of A are equal), G is strongly connected if and only if $\text{Re}(\lambda_2(\mathcal{L})) > 0$. Therefore, if a directed network is balanced, then the magnitude of the real part of $\lambda_2(\mathcal{L})$ can be used as a measure of strong connectivity [5].

In this section, we show how a left eigenvector of S can be used to determine connectivity of the graph. First, we define the notion of reducible and irreducible matrices.

Definition 6 ([21]). A matrix $M \in \mathbb{R}^{N \times N}$ is *reducible* if there exists a permutation matrix $P \in \mathbb{R}^{N \times N}$ such that $P^{-1}MP$ is

upper block triangular.

Definition 7 ([21]). A matrix $M \in \mathbb{R}^{N \times N}$ is *irreducible* if it is not reducible.

The following theorem yields a test for strong connectivity in the case of directed networks:

Theorem IV.2 ([21]). A directed graph G is strongly connected if and only if its adjacency matrix A is irreducible.

This property also extends to the matrix S :

Lemma IV.3. Consider a directed graph G with adjacency matrix A and matrix $S \in \mathbb{R}^{n \times n}$ defined by (6). If G is strongly connected, then S is irreducible.

Proof: See Appendix. ■

Since $S \in \mathbb{R}^{N \times N}$ is row-stochastic, there exists at least one left eigenvector $\gamma \in \mathbb{R}^N$ such that

$$\gamma^T = \gamma^T S. \quad (8)$$

If S is obtained from a strongly connected graph, then it is irreducible, as shown in Lemma IV.3. The Perron-Frobenius Theorem [18], [22] states that if S is irreducible then the entries of γ are strictly positive. Also, γ is unique (up to a scale factor).

However, what we require is a result in the opposite direction. Given γ , can we say anything about the connectivity of G ? One answer is provided by the following lemma.

Lemma IV.4 (Lemma 2.28 [16]). For a directed graph G with Laplacian matrix \mathcal{L} , there exists a positive vector $\omega \in \mathbb{R}^N$, $\omega > 0$ such that $\omega^T \mathcal{L} = 0$ if and only if G is a disjoint union of strongly connected subgraphs.

Lemma IV.5. Let G be a weakly connected graph with a stochastic matrix S as defined in (6). Then, S has a positive left eigenvector $\gamma > 0$ such that $\gamma^T S = \gamma^T$ if and only if G is strongly connected.

Proof: Let G be strongly connected. Then, A is irreducible by Theorem IV.2 and hence so is S by Lemma IV.3. By the Perron Frobenius Theorem, S has a unique strictly positive left eigenvector corresponding to its spectral radius. Since S is a stochastic non-negative matrix, its spectral radius is 1. Thus, there exists $\gamma > 0$ such that $\gamma^T S = \gamma^T$.

Let S have a positive left eigenvector $\gamma > 0$ such that $\gamma^T S = \gamma^T$. We can derive

$$\begin{aligned} \gamma^T S &= \gamma^T \\ \Rightarrow \gamma^T (I_N - \epsilon L) &= \gamma^T \\ \Rightarrow \gamma^T - \epsilon \gamma^T \mathcal{L} &= \gamma^T \\ \Rightarrow \gamma^T \mathcal{L} &= 0. \end{aligned} \quad (9)$$

where \mathcal{L} is the graph Laplacian of G . By Lemma IV.4, G must be a disjoint union of strongly connected graphs. This statement implies that G consists of components (disjoint subgraphs) that are strongly connected. However, since G is weakly connected, it contains only one component. Thus G must be strongly connected. ■

Corollary 1. Let G be a directed graph with graph Laplacian \mathcal{L} and stochastic matrix S . Then, G is strongly connected if and only if $\lambda_2(\mathcal{L}) \neq 0$ and there exists $\gamma \in \mathbb{R}^N$ such that $\gamma^T S = \gamma^T$ and $\gamma > 0$.

C. Measure of Connectivity

The previous subsection develops a condition that yields a binary answer to whether a directed graph (that is not necessarily balanced) is strongly connected or not. Suppose that $\lambda_2(\mathcal{L}) \neq 0$ and G is strongly connected. Then, $\min_{i \in \{1, 2, \dots, N\}} \gamma_i$ could be used to measure how close the graph is to losing strong connectivity, due to Corollary 1. In this subsection, We will show that the ratios of the elements of γ are more useful.

Let γ_i be the i^{th} component of γ . Then, we can compute

$$\mu_{ij} = \frac{\gamma_i}{\gamma_j} + \frac{\gamma_j}{\gamma_i}, \quad (10)$$

for all pairs $i, j \in V$ where $i \neq j$. Due to the Perron-Frobenius theorem, if G is strongly connected then μ_{ij} are non-zero and finite $\forall i, j \in V$. The quantities μ_{ij} are functions of the state q , and are time varying.

When G is not strongly connected, S is reducible and hence γ is not necessarily unique or strictly positive. The structure of γ for reducible stochastic matrices will enable us to use the ratio of the entries of γ for irreducible matrices to indicate which edges should be preserved.

We can introduce the following definitions

Definition 8. Two nodes i, j of a directed graph are *weakly connected* to one another if there exists a directed path from i to j **or** a directed path from j to i .

Definition 9. Two nodes i, j of a directed graph are *connected* to one another if there exists a directed path from i to j **and** a directed path from j to i .

We can define the binary relation \bar{C} such that for two nodes $i, j \in G$, $i \bar{C} j$ if and only if i and j are weakly connected. One can show that \bar{C} is an equivalence relation.

Definition 10. A *component* of a directed graph G is the set of nodes in an equivalence class of V under \bar{C} .

There are two kinds of components: weakly connected components and strongly connected components. They are defined as follows.

Definition 11. A *strongly connected component* is a component in which every pair of nodes is connected.

Definition 12. A *weakly connected component* is a component in which at least two nodes are not connected.

The components of a directed graph are effectively disjoint subgraphs of G . That is, there are no edges between two nodes located in two distinct components of a graph. We can also partition a graph based on sets of nodes that are connected to each other. This partition is achieved by defining the binary relation \vec{C} such that for two nodes $i, j \in G$, $i \vec{C} j$ if and only if i and j are connected. Again, one can check that \vec{C} is an

equivalence relation. We can define the equivalence class $[i]$ of a node $i \in V$ under the relation \overrightarrow{C} as follows

Definition 13. The equivalence class $[i]$ of a node $i \in V$ under \overrightarrow{C} consists of all nodes $j \in V$ such that $i \overrightarrow{C} j$.

Let \bar{V} be the set of all equivalence classes of V under \overrightarrow{C} . We can now define three types of equivalence classes.

Definition 14. An equivalence class $[i] \in \bar{V}$ is called a *sink* if there is no directed edge $(u, v) \in E$ such that $u \in [i], v \notin [i]$.

Definition 15. An equivalence class $[i] \in \bar{V}$ is called a *source* if there is no directed edge $(u, v) \in E$ such that $v \in [i], u \notin [i]$.

Definition 16. An equivalence class $[i] \in \bar{V}$ is called a *center* if it neither a source nor a sink.

We are now in a position to describe the nature of γ based on the types of equivalence classes defined above. Consider the following result:

Lemma IV.6. *Let G be a directed graph. Then, there exists a left eigenvector γ of the matrix S associated with the eigenvalue 1 with i^{th} element γ_i such that*

$$\begin{aligned} \gamma_i &> 0 \text{ if } [i] \text{ is a source, and} \\ \gamma_i &= 0 \text{ otherwise.} \end{aligned} \quad (11)$$

Proof: A summary of the proof is that if G is not strongly connected (and hence reducible), one can convert S into a lower-triangular block diagonal stochastic matrix, through permutation of the labels in V . Each diagonal block corresponds to an equivalence class of V under \overrightarrow{C} . The off-diagonal blocks consist of edges between nodes in distinct equivalence classes. Theorem 4.7 in [23] describes the space of left eigenvectors associated with the eigenvalue 1 of such a matrix, yielding the result. ■

We can extend Lemma IV.6 to the case of a graph with at least one weakly connected component.

Lemma IV.7. *If G is a directed graph with at least one weakly connected component, then there exists at least one equivalence class $[i] \in \bar{V}$ that is a source, and there exists at least one equivalence class $[j] \in \bar{V} \setminus \{[i]\}$ that is not a source.*

Proof: By definition, a weakly connected component must consist of more than one equivalence class under \overrightarrow{C} . Since the component is weakly connected, there must be at least one directed edge $(i, j) \in E$ such that node i or node j belong to distinct equivalence classes.

Let all these equivalence classes be sources. Then, there is no equivalence class with an in-edge. This conclusion contradicts the conclusion that there must be at least one edge between nodes in different equivalence classes. Therefore, there must be at least one equivalence class in the weakly connected component that is not a source. Due to similar reasoning, there must be at least one equivalence class that is a source. ■

D. Minimal connecting edge sets

If a graph $G = (V, E)$ is strongly connected, then $\gamma > 0$. Suppose that we delete some edges $E_0 \subset E$, such that the resulting graph $G' = (V, E - E_0)$ has at least one weakly connected component. We can use Lemma IV.7 to identify some of the edges in E_0 .

Given a directed graph $G = (V, E)$ that is not strongly connected, we can define a *minimal connecting edge set* E' as follows:

Definition 17. A *minimal connecting edge set* E' of a graph $G = (V, E)$ is a set of edges such that $(V, E \cup E')$ is strongly connected, and for any $e \in E'$, $(V, E \cup (E' - e))$ is not strongly connected.

If $G(t)$ for $t < T$ is strongly connected, and $G(T)$ has at least one weakly connected component, then at $t = T$ some set of edges E_0 were deleted from $G(t)$ where $t < T$. The set E_0 must contain a minimal connecting edge set for the graph $G(T)$. We can claim the following:

Lemma IV.8. *Let $G = (V, E)$ be a directed graph with at least one weakly connected component. For any minimal connecting edge set E' of G , there exists at least one edge $(i, j) \in E'$ such that $[j]$ is a source and $[i]$ is not a source.*

Proof: Let the set of equivalence classes of V under \overrightarrow{C} be \bar{V} . Let \bar{V}_s be the set of sources in \bar{V} , and $\bar{V}_{ns} = \bar{V} \setminus \bar{V}_s$. Since G has at least one weakly connected component, \bar{V}_s and \bar{V}_{ns} are non-empty, by Lemma IV.7. There is no edge in E starting on a node belonging to an element of \bar{V}_{ns} and ending on a node belonging to an element of \bar{V}_s .

Let E' be a minimal connecting edge set of G . This implies that $G' = (V, E \cup E')$ is strongly connected. If no edge $(i, j) \in E'$ exists such that $[i] \in \bar{V}_{ns}$ and $[j] \in \bar{V}_s$, then the same holds for the the graph G' with edge set $E \cup E'$. This conclusion implies that G' is not strongly connected, which is a contradiction. Therefore, there must be at least one edge $(i, j) \in E'$ such that $[j]$ is a source and $[i]$ is not a source. ■

Lemmas IV.7 and IV.8 together suggest a method to identify some of the edges that are critical to preserving strong connectivity in a directed graph which is close to losing it.

Lemma IV.9. *Let $G(c) = (V, E)$ be a graph parametrized continuously by $c \in \mathbb{R}$ such that $G(c)$ is strongly connected $\forall c > 0$. Let $S(c)$ be its row stochastic adjacency matrix with Perron vector $\gamma(c)$. Let $G(0)$ be the subgraph of $G(c)$ obtained when $c = 0$, such that $G(0)$ has at least one weakly connected component. Let E_0 be the subset of edges in E that are deleted when $c = 0$. Let $E' \subset E_0$ be the set of edges belonging to a minimal connecting edge set of $G(0)$. Then there exists $(i, j) \in E'$ such that $\mu_{ij} \rightarrow \infty$ as $c \rightarrow 0^+$.*

Proof: $G(c)$ is strongly connected and hence $\gamma(c) > 0$. When $c = 0$, the edges in E_0 get deleted. The resulting graph is no longer strongly connected, and has at least one weakly connected component. Let $(i, j) \in E'$ be such that $[i]$ is a sink or center and $[j]$ is a source. Such an edge exists by Lemma IV.8. From Lemma IV.7 it follows that $\gamma_i(0) = 0$ and $\gamma_j(0) > 0$. Thus, $\mu_{ij} = \infty$ for $\gamma(0)$. Since $G(c)$ is continuous in c , as $c \rightarrow 0^+$, $\mu_{ij} \rightarrow \infty$. ■

Suppose a node $i \in V$ can compute or estimate γ [19], [24]. Then, this node can compute μ_{ij} for all $i, j \in V$. It can then attempt to preserve all those edges (i, j) or (j, i) such that $j \in \mathcal{N}^{\text{out}} \cup \mathcal{N}^{\text{in}}$ and $\mu_{ij} \gg 2$. Doing so will prevent the directed graph from becoming a graph with at least one weakly connected component. The construction of such an edge-preserving control law is presented in the next section.

V. EDGE-PRESERVING CONTROL

In this section, we will consider a control law that preserves selected directed edges in a directed graph. The edge weights are monotonically decreasing functions of the distances, and thus preserving edges is equivalent to ensuring that the corresponding inter-robot distances remain within a certain bound. The distance is symmetric in terms of the robot indices. This symmetry implies that the control law is based on **undirected** graphs and graph matrices. Even so, it can be applied to control of directed graphs when an appropriate undirected graph is constructed from this directed graph.

Consider a fixed undirected graph $\overleftarrow{G} = (V, \overleftarrow{E})$. The nodes V are the N robots with configurations $q_i \in \mathbb{R}^n$, $i \in \{1, 2, \dots, N\}$ and dynamics (3). For each $(i, j) \in \overleftarrow{E}$, we define real numbers R_{ij} , lb_{ij} and ub_{ij} . The constant R_{ij} denotes the maximum allowable separation between robots i and j . In other words, at any time t , we want $d_{ij}(t) < R_{ij}$, for all edges $(i, j) \in \overleftarrow{E}$. The quantity $d_{ij}(t)$ is the Euclidean distance between the two nodes, given by

$$d_{ij}(t) = \|q_i(t) - q_j(t)\|. \quad (12)$$

A continuous control that achieves the constraint $d_{ij}(t) < R_{ij}$ when $R_{ij} = \delta$ for all $(i, j) \in \overleftarrow{E}$ was proposed in [25]. The control law is of the form

$$u_i^c = \sum_{j \in \mathcal{N}_i} k_{ij}(d_{ij}(t))(q_j(t) - q_i(t)), \quad (13)$$

where $k_{ij}(d_{ij})$ is called the ‘edge-tension’ and is given by

$$k_{ij}(d_{ij}) = \frac{2\delta - d_{ij}}{(\delta - d_{ij})^2}. \quad (14)$$

We can define the function ν_{ij} as

$$\nu_{ij}(d) = \begin{cases} 0 & \text{if } d < lb_{ij}, \\ 1 - \exp\left(1 - \left(\frac{d - lb_{ij}}{ub_{ij} - lb_{ij}}\right)^2\right) & \text{if } lb_{ij} < d < ub_{ij}, \\ 1 & \text{if } ub_{ij} < d, \end{cases} \quad (15)$$

which is a smooth monotonic function such that $\nu_{ij}(lb_{ij}) = 0$ and $\nu_{ij}(ub_{ij}) = 1$. The numbers lb_{ij} and ub_{ij} are chosen such that $0 < lb_{ij} < ub_{ij} < R_{ij}$.

We propose a modified edge tension given by

$$\bar{k}_{ij}(d_{ij}) = k_{ij}(d_{ij})\nu_{ij}(d_{ij}) \quad (16)$$

where $k_{ij}(d_{ij})$ is given by

$$k_{ij}(d_{ij}) = \frac{2R_{ij} - d_{ij}}{(R_{ij} - d_{ij})^2}. \quad (17)$$

The numbers lb_{ij} , ub_{ij} , and R_{ij} can be determined at the instant of time when it is decided that (i, j) or (j, i) must

be preserved. The procedure to choose these variables is described in Section VI. Selecting them appropriately ensures that $\bar{k}_{ij}(t) = \bar{k}_{ij}(d_{ij}(t))$ is continuous in time.

The modified edge tension results in a new control given by

$$\begin{aligned} u_i^c &= \sum_{(i,j) \in \overleftarrow{E}} \bar{k}_{ij}(d_{ij}(t))(q_j(t) - q_i(t)) \\ &= \sum_{(i,j) \in \overleftarrow{E}} k_{ij}(d_{ij}(t))\nu_{ij}(d_{ij}(t))(q_j(t) - q_i(t)). \end{aligned} \quad (18)$$

Consider the set D defined as

$$D := \{q \in \mathbb{R}^{Nn} : d_{ij} < R_{ij} \forall (i, j) \in \overleftarrow{E}\}. \quad (19)$$

Theorem V.1. Consider an undirected graph $\overleftarrow{G} = (V, \overleftarrow{E})$ with dynamics (3). Let the feedback u_i^c be selected according to (18). The external control $u_i^e(t)$ is unknown but bounded for each $i \in V$. Then, for any solution $q(t)$ of (3) with initial condition $q(t_0) \in D$,

$$q(t) \in D \forall t \geq t_0.$$

Proof: See Appendix. ■

Remark 1. The proof Theorem of V.1 relies on the fact that \bar{k}_{ij} becomes unbounded exactly when $d_{ij} = R_{ij}$, and is bounded otherwise. Thus, the actual shape of k_{ij} is irrelevant, allowing a large set of functions to be used for preserving inter-robot distances, as opposed to being restricted to the one in (17).

Note that the term $(q_j(t) - q_i(t))$ in (18) is simply the negative gradient of d_{ij} scaled by a function of d_{ij} . This structure has been used in the proof of Theorem (V.1). If the edge weights of the directed graph are differentiable functions of the positions of the robots, then the control in (18) can be extended to such directed graphs.

VI. PROPOSED CONNECTIVITY CONTROLLER

In the previous section, we have shown how to ensure that any two robots i and j remain within some desired distance R_{ij} of each other. This guarantee holds for any set of additional bounded control terms u_i^e and in spite of the fact that multiple edges may be close to reaching their respective limits on allowable separation.

In a directed graph $G = (V, E)$, suppose that the directed edge (i, j) must be preserved. In other words, $d_{ij}(t) < R_i$ must hold for all $t > t_0$, where t_0 is the initial time instant. If $(i, j) \in E$ or $(j, i) \in E$ needs to be preserved, there is some distance $R_{ij} < \infty$ such that d_{ij} must always be less than R_{ij} . Thus, (i, j) is included in the edge set \overleftarrow{E}_c as an undirected edge. We thus obtain an undirected graph $\overleftarrow{G}_c = (V, \overleftarrow{E}_c)$.

We select thresholds $\mu_{off} \in \mathbb{R}$ and $\mu_{on} \in \mathbb{R}$ such that $2 < \mu_{off} < \mu_{on}$. These thresholds serve to decide if an edge should be included in \overleftarrow{E}_c . The edge set \overleftarrow{E}_c varies with time, and can be represented using a switching adjacency matrix $\overleftarrow{A}_c = \{a_{ij}^c\} \in \mathbb{R}^{N \times N}$. We assume that at the initial condition $q(t_0)$, $\mu_{ij} < \mu_{on}$ for all $i, j \in V$, and therefore $\overleftarrow{E}_c = \emptyset$. This assumption implies that $a_{ij}^c = 0$ for all $i, j \in V$.

The elements of \overleftrightarrow{A}_c are modified using the following update rule:

$$a_{ij}^c = \begin{cases} 1 & \text{if } w_{ij} + w_{ji} > 0 \text{ and } \mu_{ij} \geq \mu_{on}, \text{ or} \\ 0 & \text{if } \mu_{ij} < \mu_{off} \text{ and } d_{ij} < lb_{ij}, \end{cases} \quad (20)$$

where μ_{ij} is obtained from (10). Naturally, \overleftrightarrow{A}_c is always a symmetric matrix. With this definition of \overleftrightarrow{A}_c , we can define \overleftrightarrow{E}_c at any time as $\overleftrightarrow{E}_c = \{(i, j): a_{ij}^c = 1\}$. At a time instant t_{sw} when a_{ij}^c switches from 0 to 1, we can set the quantities R_{ij} , lb_{ij} and ub_{ij} as follows:

$$R_{ij} = \begin{cases} \max\{R_i, R_j\} & \text{if } \min\{w_{ij}, w_{ji}\} = 0, \text{ or} \\ \min\{R_i, R_j\} & \text{if } \min\{w_{ij}, w_{ji}\} > 0, \end{cases} \quad (21a)$$

$$lb_{ij} = d_{ij} + (1 - c)(R_{ij} - d_{ij}), \text{ and} \quad (21b)$$

$$ub_{ij} = d_{ij} + c(R_{ij} - d_{ij}), \quad (21c)$$

where $c \in (0.5, 1]$. Equation (21) implies that $\bar{k}_{ij}(d_{ij}(t_{sw})) = 0$ by design. Thus, the control u_i^c is never discontinuous in time. When a_{ij}^c switches from 1 to 0, the value of these quantities are irrelevant.

Note that the entries \overleftrightarrow{A}_c are functions of time and may switch at various time instants. At each instant of switching, the quantities R_{ij} , lb_{ij} and ub_{ij} are reset based on (21). Thus, we have suppressed the fact that these quantities are piecewise constant in time, as opposed to constants for all time.

If we only use μ_{ij} to select edges \overleftrightarrow{E}_c , then the resulting closed loop system may still suffer a loss of strong connectivity. If two robots i and j have identical communication radii, then the edges (i, j) and (j, i) are identical and will be broken simultaneously. The loss of two such edges could lead to a situation where the two resulting components are each strongly connected. As $w_{ij} (= w_{ji})$ approaches zero, μ_{ij} remains finite, and so (i, j) may not get included in \overleftrightarrow{E}_c . This means that this undirected edge will not be preserved.

In order to prevent this case, we modify the update rule (20) as follows

$$a_{ij}^c : \quad (22)$$

$$= \begin{cases} 1 & \text{if } w_{ij} + w_{ji} > 0 \text{ and } (\mu_{ij} \geq \mu_{on} \text{ or } |\lambda_2(\mathcal{L})| < \lambda_m), \text{ or} \\ 0 & \text{if } (\mu_{ij} < \mu_{off} \text{ and } |\lambda_2(\mathcal{L})| > \lambda_m) \text{ and } d_{ij} < lb_{ij} \end{cases}$$

where $\lambda_m > 0$ is some threshold that determines when all edges must be included. It is possible to add a hysteresis effect for the switching based on λ_m also.

We are now ready to state the main result related to the performance of the control that preserves strong connectivity.

Theorem VI.1. *Consider a directed mobile communication network with dynamics of the i^{th} robot given by*

$$\dot{q}_i(t) = u_i^c(t) + u_i^e(t), \quad (23)$$

where $u_i^e(t)$ is a bounded vector for all t . Let $u_i^c(t)$ be given by

$$u_i^c(t) = \sum_{(i,j) \in \overleftrightarrow{E}_c} \bar{k}_{ij}(d_{ij}(t))(q_j(t) - q_i(t)), \quad (24)$$

where \overleftrightarrow{A}_c (and hence \overleftrightarrow{E}_c) is updated according to (22), parameters R_{ij} , lb_{ij} and ub_{ij} are updated according to (21),

and \bar{k}_{ij} is given by (16). If the network is strongly connected at some time t_0 , then the network is strongly connected for all $t > t_0$.

Proof: Due to Corollary 1 and Lemma IV.9, the update rule (22) ensures that the edge set \overleftrightarrow{E}_c always contains edges that must be preserved in order to maintain strong connectivity, when the network is close to losing strong connectivity. By Theorem V.1, the control (24) always ensures preservation of edges in \overleftrightarrow{E}_c . Thus, if the network is strongly connected at $t = t_0$, it is strongly connected for all $t > t_0$. ■

A. Implementation of the proposed controller

Consider a directed edge $(i, j) \in E$, with distance d_{ij} between robots i and j satisfying the inequalities $R_j < d_{ij} < R_i$. In other words, robot i can send information to robot j , but robot j cannot send information to robot i . As mentioned previously, the directed edge $(i, j) \in E$ is also treated as an undirected edge that must be included in \overleftrightarrow{E}_c . As a result of \overleftrightarrow{E}_c being undirected, the control law for each robot will depend on the positions of both its in-neighbors $\mathcal{N}_i^{\text{in}}$ and out-neighbors $\mathcal{N}_i^{\text{out}}$. In particular, $(i, j) \in \overleftrightarrow{E}_c$ implies that the control terms of both robots i and j include the term $\pm \bar{k}_{ij}(q_j - q_i)$. By assumption A1, robot i can measure d_{ij} and $q_j - q_i$, and can therefore implement a control which includes such a term.

Robot j must also compute the vector quantity $\bar{k}_{ij}(q_j - q_i)$ in order to include it in the control u_j^c . Robot j can obtain the magnitude of this vector, given by $d_{ij}\bar{k}_{ij}$, from robot i , because robot i can send information to robot j through edge $(i, j) \in E$. Robot j would also need to know the direction defined by $q_j - q_i$ in order to compute $\bar{k}_{ij}(q_j - q_i)$. Note that robot j does not need to be able to send any information to robot i . If assumption A2 holds, then robot j can measure the required direction, and then compute the required vector quantity. Unlike assumption A1, assumption A2 is not commonly required in multi-robot coordination methods.

We suggest two sensing methods that may result in the satisfaction of assumption A2 for practical systems. The first is to use radio direction finding (RDF) techniques. These techniques use a specialized receiver (different from the transmitter) to estimate the direction from which received communication signals are originating. A second method that enables satisfaction of assumption A2 for a multi-robot team is to equip all robots with omnidirectional sensors that have a spherical sensing region defined by a radius R^s , where $R^s = \max_{i \in V} R_i$. The presence of such sensors immediately implies that both robots forming a directed communication edge (i, j) can measure both direction and magnitude of the vector $\pm(q_i - q_j)$.

One may ask whether including the contribution to (24) from the in-neighbours is necessary. We argue that it is necessary, since a single agent i cannot unilaterally guarantee the preservation of an edge (i, j) when arbitrary external control terms u_j^e are present. This situation is encountered in a simulation involving 11 robots using a control of the form

$$u_i^c = \sum_{j \in \mathcal{N}_i^{\text{out}}} \mu_{ij}(q_j - q_i) \quad (25)$$

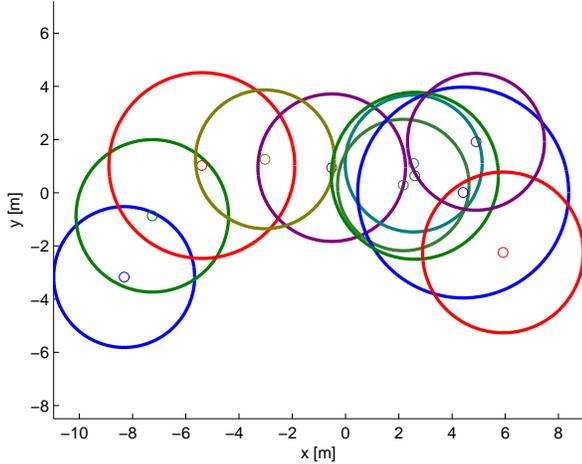


Fig. 2. Simulation of 11 robots with velocities based on an external task based control and the control (25) which is based on out-neighbors only. The configuration corresponds to the moment before loss of strong connectivity. A zoomed-in view of the critical link and nodes is presented in Figure 3.

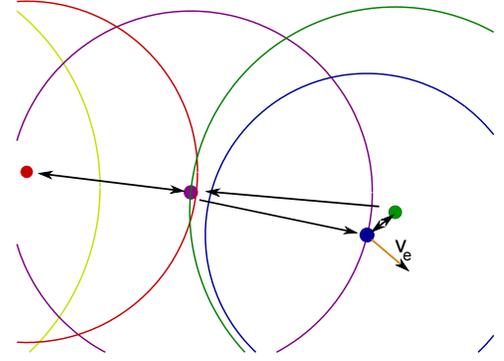


Fig. 3. Zoomed-in view of Figure 2. Note that some robots and communication boundaries have been removed, and colors changed to enhance clarity. The blue purple robot is stationary due to the weak links on either side (to the red and blue robots). The blue robot does not sense the purple robot, and hence does not assist in preservation of the critical link from the purple robot to the blue robot. V_e is the blue robot's additional task-based velocity.

along with an external input $u_i^e \in \mathbb{R}^2$.

Figure 2 shows the positions and communication radii of the 11 robots implementing (25) with some additional task-based control that is bounded on compact sets. This configuration corresponds to a moment just before the loss of strong connectivity. In Figure 3 we zoom in on the purple robot (close to the origin). Note that we change colors of the remaining robots for clarity. The critical edge is the edge from the purple to the blue robot. Due to the weak link with the red robot, the purple robot cannot move with arbitrary speed to the right. The blue robot does not sense the purple robot, and hence is free to move towards the right, due to the external task control v_e . Thus, the edge from the purple to the blue robot will be broken, and the graph is no longer strongly connected. This example demonstrates that both robots must be responsible for preserving the critical edge when disturbances exist, and hence a control law of the form (24) is proposed.

VII. SIMULATION

We simulate four robots in \mathbb{R}^2 that are commanded with velocities $u_i^c + u_i^e$ where u_i^c is given by (24), and u_i^e is some additional task-dependent velocity. The task velocities u_i^e are such that if the agents were to move according to u_i^e alone, the network would no longer be strongly connected. In fact, the agents would become isolated from one another. The task velocities are given by

$$u_1^e = \begin{bmatrix} -1.2 \\ -1.2 \end{bmatrix} - q_1, u_2^e = \begin{bmatrix} -0.6 \\ 0 \end{bmatrix}, u_3^e = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, u_4^e = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix}.$$

In Figure 4 we see that the initial condition is such that all agents are connected to all other agents. Under the action of u_i^c given by (24), the agents reach equilibrium positions as seen in Figure 5. These positions are such that the agents form a network that is barely strongly connected, as seen from Figure 6 where the agents are located close to the communication boundaries of other agents.

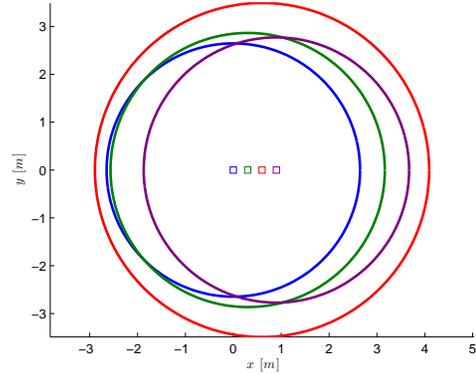


Fig. 4. Positions of the four robots at $t = 0$, denoted by squares. The boundary of the circular communication region of each robot is indicated by the circle.

A second simulation is presented where there are $N = 9$ robots. The initial condition $Q(0)$ is given by

$$Q(0) = \begin{bmatrix} 0.56 & 0.42 & 0.70 & 0.92 & 1.03 & 1.09 & 1.07 & 0.90 & 0.85 \\ 1.91 & 1.27 & 1.01 & 0.99 & 1.05 & 1.22 & 1.41 & 1.72 & 1.96 \end{bmatrix}, \quad (26)$$

where $q_i(0)$ is the i^{th} column of $Q(0)$. The radius R_i is given by the i^{th} element of the vector

$$\bar{R} = [0.83 \quad 0.55 \quad 0.31 \quad 0.18 \quad 0.28 \quad 0.26 \quad 0.42 \quad 0.50 \quad 0.41]^T. \quad (27)$$

The external control matrix $U^e(t)$ is given by

$$U^e(t) = \begin{bmatrix} -3 & 3 & 3 \sin(0.5t) & 0 & -0.6 & 0 & 0 & 0 & 0 \\ -3 & 3 & 3 \cos(0.5t) & 0 & -0.5 & 0 & 0 & 0.4 & 0 \end{bmatrix}, \quad (28)$$

where $u_i^e(t)$ is the i^{th} column of $U^e(t)$.

The initial configuration is seen in Figure 7. The graph remains strongly connected, as indicated by the non-zero values of $\min_{i \in \{1, 2, \dots, N\}} \gamma_i(t)$ and $|\lambda_2(\mathcal{L})|$ in Figure 8.

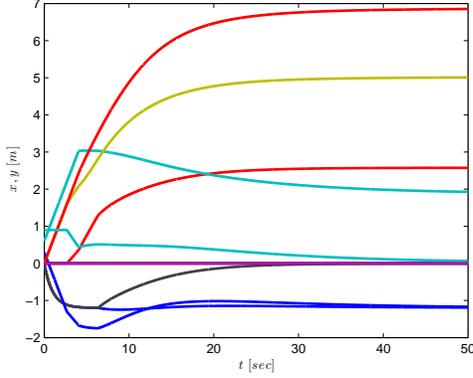


Fig. 5. Trajectories $x(t)$ and $y(t)$ for the four agents. This plot shows that the agents reach an equilibrium position.

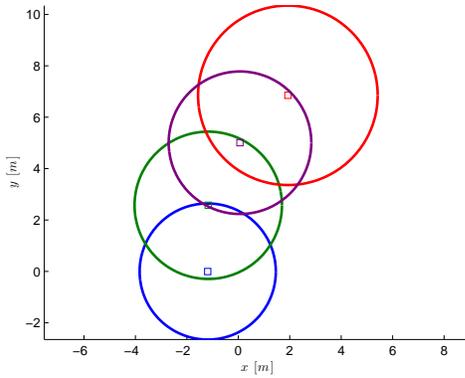


Fig. 6. Final positions of the four robots, denoted by squares. At equilibrium, the robots form a chain such that the network is still strongly connected.

VIII. CONCLUSION

In this paper, we have proposed a method to preserve distance limits between pairs of robots with single-integrator dynamics, in the presence of additional bounded external controls or disturbances. The analysis of the control law does not require a fixed closed form potential function in order to prove that the distance limits are never violated. The analysis only depends on some edge-dependent nonlinear functions possessing very general properties. This generality affords great flexibility in designing such functions. In particular, we do not need the functions to be defined in advance. This enables us to define them in a way that avoids discontinuities in the connectivity control.

The proposed edge-preservation method is combined with a method to select which edges to preserve. The method of edge selection is based on quantities derived from the Perron vector of an irreducible stochastic matrix derived from the graph, as well as the second smallest eigenvalue of the graph Laplacian. Through an analysis of the structure of the Perron vectors of reducible matrices, we can show that edges that must be preserved in order to maintain strong connectivity are always selected.

The combination of the edge-preservation method and the edge-selection method enables us to guarantee that the prox-

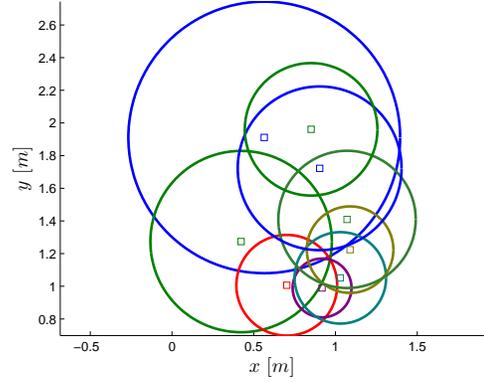


Fig. 7. Initial positions of the nine robots at $t = 0$, denoted by squares. The boundary of the circular communication region of each robot is indicated by the circles centered at the squares.

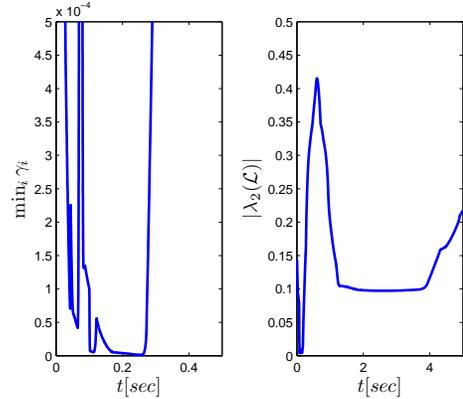


Fig. 8. Plots of $\min_{i \in \{1, 2, \dots, N\}} \gamma_i(t)$ and $\lambda_2(\mathcal{L})$. The quantity $\min_{i \in \{1, 2, \dots, N\}} \gamma_i(t)$ becomes very small but remains non-zero for $t \in [0, 0.5]$. The network is not very close to losing strong connectivity during the remaining duration of the simulation. The value of $\lambda_2(\mathcal{L})$ remains non-zero during the entire simulation.

imity graph formed by the team of robots always remains strongly connected.

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APPENDIX

A. Proof of Lemma IV.3

Proof: If the directed graph G is strongly connected, A is an irreducible matrix [21]. The matrix S is of the form $kA + B$ where k is a non-zero scalar and B is a diagonal matrix. Suppose that S is reducible. Then there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ such that $P^T S P$ is in upper block triangular form. This fact implies that $kP^T A P + P^T B P$ is upper block triangular. Since B is diagonal, so is $P^T B P$. Thus, $P^T A P$ must also be upper block triangular, which contradicts the fact that A is irreducible. Hence, S must be irreducible. ■

B. Proof of Theorem V.1

As a first step to analyzing the performance of (18), we need to rewrite the equations of the robots in such a way that the k^{th} elements of all vectors q_i are combined. This rewriting was also done in [25]. We define the matrix $Q \in \mathbb{R}^{n \times N}$ such that the i^{th} column of Q is q_i . Let x_k be the transpose of the k^{th} row of Q . Clearly, there are n such vectors. Let the state of all N robots be denoted as $q \in \mathbb{R}^{nN}$. We can define this vector in two ways. The first method consists of stacking the N vectors q_i . The second method involves stacking the n vectors x_i . We choose the second method.

Similarly, we define the matrices $U^c \in \mathbb{R}^{n \times N}$ and $U^e \in \mathbb{R}^{n \times N}$ such that the i^{th} column of U^c is exactly u_i^c and the i^{th} column of U^e is u_i^e . Then, the transpose of the i^{th} row of U^c is denoted by $u^{c,i}$ and the transpose of the i^{th} row of U^e is denoted by $u^{e,i}$.

The N equations of the form

$$\dot{q}_i = u_i^c + u_i^e \quad (29)$$

can be rewritten as the matrix equation

$$\dot{Q} = U^c + U^e, \quad (30)$$

which in turn can be written as the n equations

$$\dot{x}_i = u^{c,i} + u^{e,i}. \quad (31)$$

The main point is that solutions of (29) and (31) will be identical. The reason we want to rewrite these equations is so that the following properties can be shown:

Proposition A.1. *Given the control law u_k^c defined in (18),*

$$u^{c,k} = -\mathcal{L}_{\bar{k}} x_k,$$

where $\mathcal{L}_{\bar{k}}$ is the symmetric graph Laplacian obtained if the edge weights of edge $(i, j) \in \overleftrightarrow{E}_c$ is \bar{k}_{ij} .

Proof: The k^{th} element of u_i^c is denoted by $u_{i,k}^c$. Similarly, we can denote the k^{th} element of q_i as $q_{i,k}$. Then,

$$u_{i,k}^c = \sum_{(i,j) \in \overleftrightarrow{E}} \bar{k}_{ij}(d_{ij}(t))(q_{j,k}(t) - q_{i,k}(t)) \quad (32)$$

for each $i \in \{1, 2, \dots, N\}$. Due to the definition of x_j , $q_{i,j} = x_{j,i}$. Also, by definition of $u^{c,k}$, we have that $u_{i,k}^c = u_i^{c,k}$, where the latter is the i^{th} element of $u^{c,k}$. Thus, we can rewrite the above equation as

$$u_i^{c,k} = \sum_{(i,j) \in \overleftrightarrow{E}} \bar{k}_{ij}(d_{ij}(t))(x_{k,j}(t) - x_{k,i}(t)) \quad (33)$$

for each $i \in \{1, 2, \dots, N\}$. The right hand side of (33) is precisely the i^{th} row of the vector $\mathcal{L}_{\bar{k}} x_k$. By stacking the N scalars $u_i^{c,k}$ for each $i \in \{1, 2, \dots, N\}$, we obtain

$$u^{c,k} = -\mathcal{L}_{\bar{k}} x_k. \quad \blacksquare$$

We now define edge potential functions as

$$V_{ij}^*(q) = \int_0^{d_{ij}} s \bar{k}_{ij}(s) ds, \quad (34)$$

where we have suppressed the dependence of \bar{k}_{ij} on R_{ij} , lb_{ij} and ub_{ij} respectively. These edge potentials can be summed over all edges to obtain the overall potential function

$$V^*(q) = \sum_{(i,j) \in E} V_{ij}^*(q). \quad (35)$$

Proposition A.2. *Given the control law u_k^c defined in (18),*

$$u^{c,k} = -(\nabla_{x_k} V^*(q))^T.$$

Proof: The partial derivative of $V^*(q)$ with respect to $x_{k,i}$ (the i^{th} element of x_k) is exactly $\frac{\partial V^*(q)}{\partial q_{i,k}}$. First, note that

$$\frac{\partial V_{ij}^*(q)}{\partial d_{ij}} = d_{ij} \bar{k}_{ij} (d_{ij}) \quad (36)$$

and

$$\frac{\partial d_{ij}}{\partial q_i} = \frac{1}{d_{ij}} (q_i - q_j)^T. \quad (37)$$

Thus,

$$\frac{\partial d_{ij}}{\partial q_{i,k}} = \frac{1}{d_{ij}} (q_{i,k} - q_{j,k}), \quad (38)$$

implying that

$$\frac{\partial V_{ij}^*}{\partial q_{i,k}} = \bar{k}_{ij} (q_{i,k} - q_{j,k}). \quad (39)$$

We can compute $\frac{\partial V^*(q)}{\partial q_{i,k}}$ as follows:

$$\begin{aligned} \frac{\partial V^*(q)}{\partial q_{i,k}} &= \sum_{(i,j) \in \overleftarrow{E}} \frac{\partial V_{ij}^*}{\partial q_{i,k}} \\ &= \sum_{(i,j) \in \overleftarrow{E}} \bar{k}_{ij} (q_{i,k} - q_{j,k}) \\ &= -u_{i,k}^c. \end{aligned} \quad (40)$$

Thus, $\frac{\partial V^*(q)}{\partial x_{k,i}} = \frac{\partial V^*(q)}{\partial q_{i,k}}$ is equal to the k^{th} element of $-u_i^c$, which in turn is the i^{th} element of $-u^{c,k}$. In short, the i^{th} element of $\frac{\partial V^*(q)}{\partial x_k}$ is the i^{th} element of $-u^{c,k}$, which implies that $(\frac{\partial V^*(q)}{\partial x_k})^T = -u^{c,k}$. Equivalently, $u^{c,k} = -(\nabla_{x_k} V^*(q))^T$. ■

Clearly, Propositions A.1 and A.2 together imply that $\mathcal{L}_{\bar{k}} x_k = (\nabla_{x_k} V^*(q))^T$.

Lemma A.1. *Consider a graph $\overleftrightarrow{G} = (V, \overleftrightarrow{E})$ with dynamics (3). Let the feedback u_i^c be selected according to (18). The external control $u_i^e(t)$ is unknown but bounded for each $i \in V$. Consider the point $q^* = \{q \in \mathbb{R}^{2n} : d_{ij} = R_{ij} \forall (i,j) \in \overleftrightarrow{E}\}$. Then, for any solution $q(t)$ of (3) with initial condition $q(0) \in D$ if $q(t) \in D$ for all $t \in [t_0, T)$, then $q(T) \neq q^*$.*

Remark 2. Note that we are not yet claiming that the solution $q(t)$ remains in D . We are merely claiming that it cannot exit D by passing through a specific point given by q^* .

Proof: We will prove the Lemma by the method of contradiction. Let there be some $T > t_0$ such that $q(t) \in D \forall t \in [t_0, T)$ and $q(T) = q^*$. The value of the potential function $V^*(q)$ along the solution $q(t)$ is denoted by $V^*(t)$. Consider the function $V^*(q)$ as a Lyapunov-like function.

$V^*(q) \geq 0 \forall q \in D$. The derivative of $V^*(t)$ along solutions of (3) with control (18) is given by

$$\begin{aligned} \dot{V}^* &= \frac{\partial V^*}{\partial q} \dot{q} \\ &= \sum_{i=1}^n \frac{\partial V^*}{\partial x_i} \dot{x}_i \\ &= \sum_{i=1}^n x_i^T \mathcal{L}_{\bar{k}} \dot{x}_i \\ &= \sum_{i=1}^n x_i^T \mathcal{L}_{\bar{k}} (-\mathcal{L}_{\bar{k}} x_i + u^{e,i}) \\ &= \sum_{i=1}^n (-x_i^T \mathcal{L}_{\bar{k}}^2 x_i + x_i^T \mathcal{L}_{\bar{k}} u^{e,i}) \\ &\leq \sum_{i=1}^n (-\|\mathcal{L}_{\bar{k}} x_i\|^2 + \|\mathcal{L}_{\bar{k}} x_i\| \|u^{e,i}\|) \\ &\leq \sum_{i=1}^n -\|\mathcal{L}_{\bar{k}} x_i\| (\|\mathcal{L}_{\bar{k}} x_i\| - \|u^{e,i}\|). \end{aligned} \quad (41)$$

Note that for convenience we have dropped the dependence on time for each term in the equation above. Let $g(i, t)$ be given by

$$g(i, t) = -\|\mathcal{L}_{\bar{k}}(t) x_i(t)\| (\|\mathcal{L}_{\bar{k}}(t) x_i(t)\| - \|u^{e,i}(t)\|), \quad (42)$$

so that

$$\dot{V}^*(t) \leq \sum_{i=1}^n g(i, t). \quad (43)$$

Since the terms u_i^e are bounded, we can bound the norms $u^{e,i}$ by $M' < \infty$. The bound on $u^{e,i}$ implies $g(i, t) < 2(M')^2$ for all $i \in \{1, 2, \dots, n\}$ and $t \geq t_0$. In turn $\sum_{i=1}^k g(i, t) < 2k(M')^2$ for any $k \in \mathbb{N}$. If $\|\mathcal{L}_{\bar{k}} x_i\| > M'$ for any $i \in \{1, 2, \dots, n\}$ then $g(i, t) < 0$. Thus, in order to show that \dot{V} becomes negative, it is sufficient to show that there is at least one index i^* such that $\lim_{t \rightarrow T} g(i^*, t) < -2(n-1)(M')^2$.

Let $P \in \mathbb{R}^{N \times N-1}$ be any matrix such that $P^T \mathbf{1}_N = 0$ and $P^T P = I_{N-1}$, where I_{N-1} is the identity matrix of size $N-1$. Let $x \in \mathbb{R}^N$ be any vector. We can define $x^P = P^T x$, and thus express x as

$$x = P x^P + \left(\frac{x^T \mathbf{1}_N}{N} \right) \mathbf{1}_N. \quad (44)$$

Given a symmetric graph Laplacian $\mathcal{L} \in \mathbb{R}^{N \times N}$ and the vector x , we have that

$$\|\mathcal{L} x\| \geq \lambda_2(\mathcal{L}) \|x^P\|. \quad (45)$$

We have assumed that $q(T) = q^*$, implying that for each edge $(i, j) \in \overleftrightarrow{E}$, we have that $\lim_{t \rightarrow T} d_{ij}(t) = R_{ij}$. One can show that $\|x_k^P(T)\| \geq \frac{R_{ij}}{\sqrt{2n}} > 0$ for at least one index $k \in \{1, 2, \dots, n\}$, where $x_k^P(T) = P^T x_k(T)$. Let this index be i^* .

Thus,

$$\begin{aligned} \|\mathcal{L}_{\bar{k}} x_{i^*}\| &\geq \lambda_2(\mathcal{L}_{\bar{k}}) \|x_{i^*}^P\| \\ &\geq \lambda_2(\mathcal{L}_{\bar{k}}) \frac{R_{ij}}{\sqrt{2n}}. \end{aligned} \quad (46)$$

We assume that \overleftrightarrow{G} is connected, which implies that $\lambda_2(\mathcal{L}_{\bar{k}}) > 0$. For any undirected graph \overleftrightarrow{G} , $\lambda_2(\mathcal{L}(\overleftrightarrow{G}))$ is a non-decreasing function of the edge weights [9]. Consider the weighted symmetric graph Laplacian $\mathcal{L}_{\bar{k},min}$ where every edge weight \bar{k}_{ij} is replaced by $\bar{k}_{min} = \min_{(i,j) \in \overleftrightarrow{E}} \bar{k}_{ij}$. Now, this matrix is converted to $\mathcal{L}_{\bar{k}}$ by *increasing* each edge weight. This fact means that

$$\lambda_2(\mathcal{L}_{\bar{k}}) \geq \lambda_2(\mathcal{L}_{\bar{k},min}). \quad (47)$$

However,

$$\begin{aligned} \lambda_2(\mathcal{L}_{\bar{k},min}) &= \lambda_2(\bar{k}_{min}\mathcal{L}_1) \\ &= \bar{k}_{min}\lambda_2(\mathcal{L}_1), \end{aligned} \quad (48)$$

where \mathcal{L}_1 is the graph Laplacian of \overleftrightarrow{G} obtained when edge weights are either zero or one. Thus, we can conclude that

$$\lambda_2(\mathcal{L}_{\bar{k}}) \geq \bar{k}_{min}\lambda_2(\mathcal{L}_1), \quad (49)$$

where $\lambda_2(\mathcal{L}_1) > 0$ since \overleftrightarrow{G} is connected. Due to the form of the edge weights, for all edges $(i,j) \in \overleftrightarrow{E}$ we have that $\bar{k}_{ij}(t) \rightarrow \infty$ as $t \rightarrow T$. Since there are a finite number of edges, it holds that $\lim_{t \rightarrow T} \bar{k}_{min}(t) \rightarrow \infty$.

From (46) and (49), we can conclude that as $t \rightarrow T$, the term $\|\mathcal{L}_{\bar{k}}x_{i^*}(t)\|$ becomes unbounded. Then, $\lim_{t \rightarrow T} g(i^*, t) = -\infty$, implying that there exists $\tau < T$ such that $g(i^*, t) < -2(n-1)(M')^2 \forall t \in [\tau, T]$. Thus, $\dot{V}^* \leq 0 \forall t \in [\tau, T]$.

This conclusion contradicts the assumption that $q(T) = q^*$, since $V^*(q^*)$ is unbounded and clearly $V^*(t) \leq V^*(\tau) \forall t \geq \tau$. Thus, it is not possible for all the edges to be disconnected at exactly the same instant. ■

Theorem V.1

Proof: Again, we use the method of proof by contradiction. Let there be some $T > t_0$ where $d_{ij}(T) = R_{ij}$ for all edges $(i,j) \in \overleftrightarrow{E}_{br,T} \subset \overleftrightarrow{E}$, where $\overleftrightarrow{E}_{br,T} \neq \emptyset$. If $\overleftrightarrow{E}_{br,T} = \overleftrightarrow{E}$ then the analysis of the previous Lemma shows that this situation cannot occur. Thus, let $\overleftrightarrow{E} - \overleftrightarrow{E}_{br,T} \neq \emptyset$.

By definition of $\overleftrightarrow{E}_{br,T}$, it must hold that for all $(i,j) \in \overleftrightarrow{E} - \overleftrightarrow{E}_{br,T}$, $d_{ij}(T) < R_{ij}$. Due to the definition of $\bar{k}_{ij}(t)$, it must hold that $\bar{k}_{ij}(T) < M_T \forall (i,j) \in \overleftrightarrow{E} - \overleftrightarrow{E}_{br,T}$ and some positive number $M_T \in \mathbb{R}$.

Define the set $V_{br,T}$ as

$$V_{br,T} = \left\{ i \in V : (i,j) \in \overleftrightarrow{E}_{br,T} \text{ for some } j \in V \setminus \{i\} \right\}. \quad (50)$$

Now, for any node $i \in V_{br,T}$, we can partition its neighbor set \mathcal{N}_i derived from \overleftrightarrow{E} into two neighbor sets \mathcal{N}_i^{br} and $\mathcal{N}_i' = \mathcal{N}_i - \mathcal{N}_i^{br}$ where the former is derived from the edge set $\overleftrightarrow{E}_{br,T}$. The control u_i^c can be partitioned into two terms based on these two disjoint neighbor sets. That is,

$$\begin{aligned} u_i^c &= \sum_{j \in \mathcal{N}_i^{br}} \bar{k}_{ij}(t)(q_j(t) - q_i(t)) + \sum_{j \in \mathcal{N}_i'} \bar{k}_{ij}(t)(q_j(t) - q_i(t)) \\ &= u_{i,br}^c + u_i^d. \end{aligned}$$

Thus, for any node $i \in V_{br,T}$, we have dynamics

$$\dot{q}_i = u_i^c + u_i^e = u_{i,br}^c + u_i^d + u_i^e \quad (51)$$

$$= \bar{u}_i^c + \bar{u}_i^e \quad (52)$$

where $\bar{u}_i^c = u_{i,br}^c$ and $\bar{u}_i^e = u_i^d + u_i^e$. The main point is that $\|\bar{u}_i^e\|$ is bounded, since for any $j \in \mathcal{N}_i'$, $(i,j) \in \overleftrightarrow{E} - \overleftrightarrow{E}_{br,T}$. Thus, \bar{u}_i^e is also bounded.

The key idea is that the subgraph $(V_{br,T}, \overleftrightarrow{E}_{br,T})$ has dynamics (52) with control \bar{u}_i^c of the form (18) where the neighbors of $i \in V_{br,T}$ are determined by the edge set $\overleftrightarrow{E}_{br,T}$, not \overleftrightarrow{E} . The term \bar{u}_i^e is bounded. The control for robot i has not changed, we have merely relabeled different terms in the control.

We thus have a system that satisfies the conditions of Lemma A.1, when the graph in question is $(V_{br,T}, \overleftrightarrow{E}_{br,T})$ with control law \bar{u}_i^c and external bounded term \bar{u}_i^e . Lemma A.1 states that there cannot be some time T where $d_{ij}(T) = R_{ij}$ for all edges $(i,j) \in \overleftrightarrow{E}_{br,T}$, assuming $d_{ij}(t) < R_{ij}$ for $t < T$. However, this conclusion violates our assumption that $d_{ij}(T) = R_{ij}$ for all edges $(i,j) \in E_{br,T}$. This contradiction leads to the conclusion that $q(t) \in D \forall t \geq t_0$. ■

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