

Maintaining Visibility in Multi-Robot-Networks with Limited Field-of-View Sensors

Hasan A Poonawala and Mark W Spong

Abstract—A multi-robot team can be endowed with a sensing graph whose edges indicate the ability of a robot to measure the relative position to another robot. The connectivity properties of this graph are important for the performance of different multi-robot coordination algorithms. Many coordination methods assume that the sensing graph is undirected. When the robots are equipped with limited field-of-view sensors, the sensing graph becomes directed. This paper provides a method to maintain the strong connectivity of a directed sensing graph in the presence of bounded persistent disturbances.

I. INTRODUCTION

The consensus algorithm [1], [2] is often used to design algorithms for the coordination of multiple robots. Information consensus can be used by a team of robots to provide all members of the team with a global average of some quantity through only local exchange of information. In order for each robot's local variable to converge to the true global average, it is necessary for the graph to retain certain connectivity properties. When the robots are mobile, the required connectivity properties may be lost.

Maintaining the connectivity of dynamic networks formed by mobile robots or agents has received a lot of attention [3]–[9]. Most solutions focus on the case of undirected graphs. Few solutions exist for the case of directed graphs [9]–[11]. Those that do exist focus on maintaining the communication network formed by multiple robots that have omnidirectional sensing and communication capabilities. The communication network may be directed when the robots have different communication radii.

Many robots are equipped with sensing devices such as cameras which have a limited field-of-view. These robots cannot necessarily measure the relative position of all robots within their communication radii. Therefore, they cannot implement the connectivity preserving techniques proposed in many works, whether the communication radii are equal or not. Even if a robot can communicate with other robots, it can only measure the distance to those robots if they are within its field-of-view. If all communication radii are equal, the communication links are bi-directional. In this case, a robot that is inside the communication range of another robot but not inside the latter's field-of-view can measure the relevant inter-robot distance and provide it via communication. If the

communication radii are unequal, this may not be possible for some pairs of robots.

These considerations show that the graph related to sensing must be treated differently from the graph related to communication, and steps must be taken to preserve the connectivity properties of the sensing graph. The sensing graph needs to be strongly connected in order to be able to preserve a set of edges which ensure that the communication network is strongly connected. If the communication radii are all equal, then it may be possible to relax the required connectivity from the sensing graph being strongly connected to the sensing graph containing a directed rooted spanning tree.

A. Related Work

Maintaining the strong connectivity of the sensing network involving limited field-of-view sensors has been studied in [10], [11]. In [10], the connectivity control consists of spinning the camera with a large enough angular velocity, effectively turning the limited field-of-view camera into an omnidirectional sensor with low sampling rate. In [11] a combinatorial approach is taken. The authors note that the estimation of minimum spanning subgraphs is NP-hard, and hence suboptimal spanning subgraphs are obtained. No control method is proposed. The work in [9] is related to preserving strong connectivity in directed communication networks. The authors aim to artificially balance the graph, and then extend the gradient-based control in [6] to keep the real part of the algebraic connectivity $\lambda_2(\mathcal{L})$ non-zero, thus guaranteeing preservation of strong connectivity. The work assumes that each robot is responsible for maintaining only its out-edges, and not the in-edges. As a result, in certain cases edges can only be preserved by allowing the agents to increase their communication radius to arbitrary values. A centralized method to ensure that a rooted spanning tree exists in the sensing graph was proposed in [12]. This thesis also proposed a decentralized method to preserve the strong connectivity of communication networks. In such networks, the robots have disk-like models of communication, with unequal communication radii. A control law was proposed to maintain the strong connectivity of such directed communication graphs, which was based on preserving inter-robot distances. A stronger result is presented in [13].

B. Contributions

In this paper, we propose a decentralized hybrid control method to preserve the strong connectivity property of the

Hasan A Poonawala is with the Institute for Computational Engineering & Science, University of Texas at Austin, Austin, Texas, USA hasanp@utexas.edu

Mark W Spong is the Dean of the Erik Jonsson School of Engineering and Computer Science, University of Texas at Dallas, Richardson, Texas, USA mspong@utdallas.edu

directed sensing graph formed by robots equipped with limited field-of-view sensors. The method is based on a novel technique proposed in [12], [13] for identifying which edges must be preserved in order to preserve strong connectivity of a directed graph. A control law is presented in this paper guarantees that desired sensing edges are preserved. We prove that the proposed control method guarantees that the sensing network will remain strongly connected. The performance is demonstrated in a simulation example. The form of the control law and the analysis of the control law presented in this paper are extensions of those in [13].

II. PRELIMINARIES

A. Directed Graphs

A directed graph $G = (V, E)$ is a tuple consisting of a set of vertices V (also called nodes) and a set of edges E (also called links). If there are N nodes, then we can label them using integers from 1 to N , so that V can be identified with the set $\{1, 2, \dots, N\}$. An edge $e \in E$ is an ordered pair (i, j) which indicates that a connection exists which starts at node i and ends at node j , where $i, j \in V$. More precisely, the edge (i, j) is an out-edge for node i and an in-edge for node j .

We can define two neighbor sets for each node $i \in V$. These are the set of out-neighbors denoted by $\mathcal{N}_i^{\text{out}}$ and the set of in-neighbors denoted by $\mathcal{N}_i^{\text{in}}$. They are given by

$$\mathcal{N}_i^{\text{out}} = \{j \in V | (i, j) \in E\} \quad (1)$$

$$\mathcal{N}_i^{\text{in}} = \{j \in V | (j, i) \in E\} \quad (2)$$

Theorem II.1. *A path is a sequence of edges such that if an edge in the path ends at node i , then the next edge in the sequence starts at node i .*

Directed networks have multiple definitions of connectivity [14]:

Definition 1. A directed graph G is *weakly connected* if a path exists from any node to any other node when the path disregards the direction of edges.

Definition 2. A directed graph $G = (V, E)$ is *quasi-strongly connected* if there exists a node $v \in V$ such that a directed path exists from any node $u \in V$ to v .

Definition 3. A directed graph G is *strongly connected* if a directed path exists from any node to any other node.

B. Mobile Robot Networks

Consider a team of N mobile robots. The configuration of the i^{th} robot is given by $q_i = (x_i, y_i, \theta_i) \in \mathbb{R}^3$. The angle θ_i is the orientation of the limited field-of-view sensor of the i^{th} robot, not the robot heading. The robot is assumed to be holonomic. We can stack the N configurations of the N robots together in an obvious way to obtain the configuration of the team $q \in \mathbb{R}^{3N}$.

The robots are assumed to have first order dynamics:

$$\dot{q}_i = u_i^c + u_i^e \quad (3)$$

where $u_i^c \in \mathbb{R}^3$ is a control term to be designed and $u_i^e \in \mathbb{R}^3$ is a bounded vector representing additional control objectives and/or disturbances.

We wish to model the directed sensing network G formed by this team of mobile robots. The nodes V of the network are the robots, and edges E in the network correspond to the ability of the robots to measure the state of other robots to which they can send information. Robot i is able to send information to any other robot within a distance of R_i from the center of the robot.

The distance d_{ij} between the i^{th} and j^{th} robot is $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$. If $d_{ij} < R_i$, then robot i can send information to robot j . Robot j is in the field of view of robot i if $\psi_{ij} < \psi_i^{\text{max}}$, where

$$\psi_{ij} = \tan^{-1} \left(\frac{y_j - y_i}{x_j - x_i} \right) - \theta_i. \quad (4)$$

We define the edge set E as

$$E = \{(i, j) : i, j \in V, d_{ij} < R_i \text{ and } \psi_{ij} < \psi_i^{\text{max}}\} \quad (5)$$

Note that the edges of the sensing graph are a subset of the edges of the communication graph, since an edge (i, j) is present in the communication graph if and only if $d_{ij} < R_i$. We are not concerned with the communication graph in the rest of this paper, since the connectivity properties of the communication graph can only be better than that of the sensing graph.

We make two assumptions about the sensing and communication abilities of these robots:

- A1 The i^{th} robot can measure the relative location of any robot inside its field-of-view.
- A2 The communication devices are such that if robot i receives information from robot j , then it can estimate the direction towards j .

In this way, the configuration q of the mobile robot team and the parameters R_i and ψ_i^{max} , $i \in \{1, 2, \dots, N\}$ determine the proximity graph G . The graph G is not a fixed graph, but rather it depends on the time-varying state $q(t)$, and hence the graph is actually dynamic and state-dependent.

III. ALGEBRAIC CONNECTIVITY

In this section, we describe concepts from algebraic graph theory that will be used in designing the connectivity-preserving control law. In Section III-A, we define the weights of the sensing graph, and introduce relevant graph matrices. In Section III-B, we recount ideas from [12], [13] which provide a method to identify which edges of a directed sensing network are critical to preserving strong connectivity.

A. Graph Matrices

Given a directed graph $G = (V, E)$, we can assign a weight w_{ij} to each edge $(j, i) \in E$. Once the edge weights w_{ij} are defined, the adjacency matrix $A_w(G) = \{a_{ij}\} \in \mathbb{R}^{N \times N}$ is given by

$$a_{ij} = \begin{cases} w_{ij} & \text{if } (j, i) \in E \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

Note that the non-zero entries of the i^{th} row correspond to the edges directed towards node i , which are the in-edges of i . It is also possible to define the i^{th} row based on out-edges, however we do not use that formulation.

Consider the smooth monotonic map Ψ given by

$$\Psi_{(\rho_1, \rho_2)}(x) = \begin{cases} 1 & \text{if } x \leq \rho_1 \\ \frac{\exp(-\frac{1}{\rho_2-x})}{\exp(-\frac{1}{\rho_2-x}) + \exp(\frac{1}{\rho_1-x})} & \text{if } \rho_1 < x < \rho_2 \\ 0 & \text{if } \rho_2 \leq x \end{cases} \quad (7)$$

The edge weights w_{ij} for the graph G are determined by d_{ij} and ψ_{ij} as follows

$$w_{ij} = \Psi_{(0, \psi_i^{max})}(|\psi_{ij}|) \Psi_{(0, R_i)}(d_{ij}) \quad (8)$$

The Laplacian $\mathcal{L}_w(G) \in \mathbb{R}^{N \times N}$ of the graph can be derived from the adjacency matrix $A_w(G)$ and is given by

$$\mathcal{L}_w(G) = D_w(G) - A_w(G) \quad (9)$$

where $D_w(G)$ is a diagonal matrix whose i^{th} diagonal element is $\sum_{j=1}^N w_{ij}$. Due to the definition of A_w , the matrix \mathcal{L}_w is often known as the in-Laplacian, however we do not refer to it as such in the paper. The Laplacian $\mathcal{L}_w(G)$ always has an eigenvalue at 0, corresponding to a right eigenvector given by $\frac{1}{\sqrt{N}} \mathbf{1}_N$, where $\mathbf{1}_N \in \mathbb{R}^N$, $\mathbf{1}_N = [1, \dots, 1]^T$.

The remaining eigenvalues of \mathcal{L}_w may be complex, and are ordered based on their absolute value. Let the i^{th} smallest eigenvalue of a matrix A be denoted by $\lambda_i(A)$. The second smallest eigenvalue of $\mathcal{L}_w(G)$ is denoted by $\lambda_2(\mathcal{L})$. In the case of undirected graphs, it is called the Fiedler value of the graph [15]. Note that we will henceforth drop the subscript w from these matrices, with the understanding that a graph matrix with no subscript implies that the edge weights correspond to w_{ij} given by (8).

We can create a nonnegative row-stochastic [16] matrix $S \in \mathbb{R}^{N \times N}$, by using the transformation

$$S = I_N - \varepsilon \mathcal{L} \quad (10)$$

where I_N is the identity matrix of size N and $\varepsilon > 0$ is a sufficiently small number. Assuming that each edge weight w_{ij} is bounded above by 1, then selecting $\varepsilon \leq 1/N$ ensures S is non-negative.

B. Algebraic Connectivity for Directed Graphs

The magnitude of $\lambda_2(\mathcal{L})$ indicates whether the network is quasi-strongly connected (and therefore weakly connected) or not. This is due to the following theorem derived from results in [17]:

Theorem III.1 ([17]). *Consider a directed graph G with Laplacian as defined in (9). The graph G is quasi-strongly connected if and only if $|\lambda_2(\mathcal{L})| > 0$*

For a directed graph, strong connectivity is indicated by $\lambda_2(\mathcal{L})$ and the first left eigenvector of the matrix S . The following is a corollary to Lemma IV.5 in [13].

Corollary 1. *Let G be a directed graph with graph Laplacian \mathcal{L} and stochastic matrix S . Then, G is strongly connected if*

and only if $\lambda_2(\mathcal{L}) \neq 0$ and there exists $\gamma \in \mathbb{R}^N$ such that $\gamma^T S = \gamma^T$ and $\gamma > 0$.

Let γ_i be the i^{th} component of γ . Then, we can compute

$$\mu_{ij} = \frac{\gamma_i}{\gamma_j} + \frac{\gamma_j}{\gamma_i} \quad (11)$$

for all $i \neq j$ and $i, j \in V$. Due to the Perron-Frobenius theorem, if G is strongly connected then μ_{ij} are non-zero and finite $\forall i, j \in V$. The quantities μ_{ij} are functions of the state q , and are time varying.

Given a directed graph $G = (V, E)$ which is not strongly connected, we can define a minimal connecting edge set E' as follows:

Definition 4. A minimal connecting edge set E' of a graph $G = (V, E)$ is a set of edges such that $(V, E \cup E')$ is strongly connected, and for any $e \in E'$, $(V, E \cup (E' - e))$ is not strongly connected.

If a graph $G = (V, E)$ is strongly connected, then $\gamma > 0$ and is unique. An analysis of the structure of the eigenvectors associated with the eigenvalue 1 for a stochastic matrix S which is *not* strongly connected leads to the following result. A weaker version is presented in [12].

Lemma III.2. [13] *Let $G(c) = (V, E)$ be a graph parametrized continuously by $c \in \mathbb{R}$ such that $G(c)$ is strongly connected $\forall c > 0$. Let $S(c)$ be its row stochastic adjacency matrix with Perron vector $\gamma(c)$. Let $G(0)$ be the subgraph of $G(c)$ obtained when $c = 0$, such that $G(0)$ has at least one weakly connected component. Let E_0 be the subset of edges in E that are deleted when $c = 0$. Let $E' \subset E_0$ be the set of edges belonging to a minimal connecting edge set of $G(0)$. Then there exists $(i, j) \in E'$ such that $\mu_{ij} \rightarrow \infty$ as $c \rightarrow 0^+$.*

It is possible for all nodes $i \in V$ to compute or estimate γ [18], [19]. In the next section, we recount how γ can be used to determine which edges should be preserved, and present a control law which preserves these edges.

IV. VISIBILITY CONTROL

We use a hybrid control approach to design the connectivity-preserving control. When a strongly connected graph $G = (V, E)$ is close to losing an edge which results in a loss of strong connectivity, either $\lambda_2(\mathcal{L}) \rightarrow 0$ or $\mu_{ij} \rightarrow \infty$ for some $(i, j) \in E$. These quantities are used to drive a two-state finite-state machine associated with each edge which determines if an edge should be preserved. If an edge (i, j) is to be preserved, we include appropriate control terms in the control for robots i and j which guarantee that the edge will be preserved. The edge is preserved even if bounded disturbance velocities act on the robots.

We define parameters μ_{on} , μ_{off} and λ_m that serve as guards which trigger a change in state of the finite state machine (FSM). The state of the FSM associated with edge (i, j) is denoted as a_{ij}^c , which can have two values: 0 or 1. The rules for transitioning between states are given by

$$a_{ij}^c \rightarrow 1, \quad \text{if } w_{ij} > 0 \text{ and } \mu_{ij} \geq \mu_{on} \text{ or } |\lambda_2(\mathcal{L})| < \lambda_m \quad (12)$$

$$a_{ij}^c \rightarrow 0,$$

$$\text{if } \mu_{ij} < \mu_{off} \text{ and } |\lambda_2(\mathcal{L})| > \lambda_m \text{ and } d_{ij} < \delta_{ij} \text{ and } |\psi_{ij}| < \zeta_{ij},$$

where $\lambda_m > 0$ is some threshold that determines when all edges must be included, and δ_{ij} and ζ_{ij} are parameters which are defined later. We must select $\mu_{on} > \mu_{off} > 2$, in order to achieve a hysteresis effect for switching based on μ_{ij} . It is possible to add a hysteresis effect for the switching based on λ_m also.

The subgraph of G which must be preserved is denoted as $G_c = (V, E_c)$. The edge set $E_c \subseteq E$ is defined as $E_c = \{(i, j) \in E: a_{ij}^c = 1\}$. The edge set E_c varies with time. We assume that at the initial condition $q(t_0)$, $\mu_{ij} < \mu_{on}$ for all $i, j \in V$, and therefore $E_c = \emptyset$. This implies that a_{ij}^c is initially set as 0 for all $i, j \in V$.

The definition of edges in G (and therefore G_c) relies on d_{ij} and ψ_{ij} being below corresponding bounds for edge $(i, j) \in E$. This leads to the use of edge tension functions [20] to ensure that these quantities never exceed those bounds. However, we modify the edge tensions presented in [20] by allowing them to be defined based on the values of d_{ij} and ψ_{ij} at the instant when (i, j) is included in E_c . This allows us to avoid the discontinuities that are faced by the method in [20] whenever new edges are included in the set of preserved distances.

The control for the i^{th} robot is given by

$$u_i^c = - \sum_{\{j:(i,j)\text{or}(j,i) \in E_c\}} \left(\alpha_{ij}(d_{ij}(t)) d_{ij} \frac{\partial d_{ij}}{\partial q_i} + \beta_{ij}(\psi_{ij}(t)) \psi_{ij} \frac{\partial \psi_{ij}}{\partial q_i} \right). \quad (13)$$

where α_{ij} and β_{ij} are the edge tension functions which ensure that d_{ij} and $|\psi_{ij}|$ respectively remain within required bounds. First, we define the function ν as

$$\nu_{\rho_3, \rho_4}(x) = \begin{cases} 0 & \text{if } x < \rho_3 \\ 1 - \exp\left(1 - \left(\frac{x - \rho_3}{\rho_4 - \rho_3}\right)^2\right) & \text{if } \rho_3 < x < \rho_4 \\ 1 & \text{if } \rho_4 < x \end{cases} \quad (14)$$

which is a smooth monotonic function such that $\nu_{\rho_3, \rho_4}(\rho_3) = 0$ and $\nu_{\rho_3, \rho_4}(\rho_4) = 1$. The numbers ρ_3 and ρ_4 are chosen such that $0 < \rho_3 < \rho_4$.

Next, we define the function κ as

$$\kappa_{\rho_5}(x) = \frac{1}{(\rho_5 - x)^2}. \quad (15)$$

The edge tension functions α_{ij} and β_{ij} are then

$$\alpha_{ij}(d_{ij}) = \kappa_{R_{ij}}(d_{ij}) \nu_{\delta_{ij}, \epsilon_{ij}}(d_{ij}), \quad \text{and} \quad (16)$$

$$\beta_{ij}(\psi_{ij}) = \kappa_{\psi_i^{max}}(|\psi_{ij}|) \nu_{\zeta_{ij}, \epsilon_{ij}}(|\psi_{ij}|), \quad (17)$$

where R_{ij} , δ_{ij} , ϵ_{ij} , ζ_{ij} , and η_{ij} are parameters which are set at the time instant t_{sw} when a_{ij}^c switches from 0 to 1. These quantities are determined as follows:

$$R_{ij} = \begin{cases} \max\{R_i, R_j\} & \text{if } \min\{w_{ij}(t_{sw}), w_{ji}(t_{sw})\} = 0 \\ \min\{R_i, R_j\} & \text{if } \min\{w_{ij}(t_{sw}), w_{ji}(t_{sw})\} > 0 \end{cases} \quad (18a)$$

$$\delta_{ij} = d_{ij}(t_{sw}) + (1 - c)(R_{ij} - d_{ij}(t_{sw})) \quad (18b)$$

$$\epsilon_{ij} = d_{ij}(t_{sw}) + c(R_{ij} - d_{ij}(t_{sw})) \quad (18c)$$

$$\zeta_{ij} = |\psi_{ij}(t_{sw})| + (1 - c)(\psi_i^{max} - |\psi_{ij}(t_{sw})|) \quad (18d)$$

$$\eta_{ij} = |\psi_{ij}(t_{sw})| + c(\psi_i^{max} - |\psi_{ij}(t_{sw})|). \quad (18e)$$

where $c \in (0.5, 1]$. The values set by (18) imply that $\alpha_{ij}(d_{ij}(t_{sw})) = \beta_{ij}(|\psi_{ij}(t_{sw})|) = 0$ by design. Thus, the control $u_i^c(t)$ is never discontinuous in time. When a_{ij}^c switches from 1 to 0, the value of these quantities are zero since ν_{ij} will be zero in both edge tension functions.

Note that in [20], a function similar to κ alone is used as the edge tension, and is based on the gradient of a potential function which must be fixed for all time in order to facilitate analysis. We avoid this constraint on the potential function by choosing a different analysis technique, allowing the edge tension to be defined in a more flexible manner.

For a graph G_c , we define the set D as

$$D = \{q \in \mathbb{R}^{3N} : d_{ij} < R_{ij} \text{ and } |\psi_{ij}| < \psi_i^{max} \quad \forall (i, j) \in E_c\}. \quad (19)$$

Theorem IV.1. Consider the fixed directed graph $G_c = (V, E_c)$ with dynamics (3). Let the control u_i^c be selected according to (13). The external control $u_i^c(t)$ is unknown but bounded for each $i \in V$. Then, for any solution $q(t)$ of (3) with initial condition $q(t_0) \in D$,

$$q(t) \in D \quad \forall t \geq t_0.$$

Proof: See appendix. ■

We are now ready to state the main result related to the performance of the control that preserves strong connectivity.

Theorem IV.2. Consider a directed mobile communication network with dynamics of the i^{th} robot given by (3). The control u_i^c is selected according to (13). The external control u_i^c is unknown but bounded for each $i \in V$. The discrete states a_{ij}^c (and hence E_c) are updated according to (12) and the parameters R_{ij} , δ_{ij} , ϵ_{ij} , ζ_{ij} , and η_{ij} are updated according to (18). Then, if the network is strongly connected at some time t_0 , the network is strongly connected for all $t > t_0$.

Proof: Due to Corollary 1 and Lemma III.2, the update rule (12) ensures that the edge set E_c always contains edges that must be preserved in order to maintain strong connectivity, when the network is close to losing strong connectivity. By Theorem IV.1, the control (13) ensures preservation of edges in E_c . Thus, if the network is strongly connected at $t = t_0$, it is strongly connected for all $t > t_0$. ■

A. Implementation of the proposed controller

The definition of u_i^c in (13) makes both edges responsible for preserving a directed edge (i, j) . In order for both robots i and j attempting to preserve the directed edge (i, j) , the position vector from i to j or vice versa must be known to

both robots. Due to assumption A1, robot i can measure this quantity since robot j is in its field-of-view. Assumption A1 is a common one in mobile ad-hoc networks. It ensures that a robot can determine the position vector to its out-neighbor, using sensing.

However, robot i may not be in the field-of-view of robot j . The position vector from j to i can be determined by robot j if Assumption A2 holds. Assumption A2 is not common. It implies that a robot can obtain the direction to an in-neighbor. This assumption can be realistically met using radio direction finding techniques (RDF). This would require specialized hardware in addition to the assumed omnidirectional wireless transmitters. The distance to an in-neighbor can be obtained by communication from the in-neighbor, which can sense this quantity. Therefore, using a combination of sensing, communication, and techniques such as RDF, both robots forming a directed edge can implement the proposed control (13). However, this does not mean that the communication is bidirectional, since the RDF equipment is used for receiving a signal, not for transmitting one.

Finally, decentralized methods to estimate $\lambda_2(\mathcal{L})$ and γ have been presented in [19] and [18] respectively.

V. SIMULATION

We simulate four robots with position in \mathbb{R}^2 and an orientation which are commanded with velocities $u_i^c + u_i^e$ where u_i^c is given by (13), and u_i^e is some additional task-dependent velocity.

The initial condition $Q(0)$ is given by

$$\begin{aligned} q_1(0) &= [0.56m \quad 1.91m \quad 4.71rad]^T \\ q_2(0) &= [0.42m \quad 1.27m \quad 0rad]^T \\ q_3(0) &= [0.70m \quad 1.01m \quad 0rad]^T \\ q_4(0) &= [0.92m \quad 0.99m \quad 2.36rad]^T \end{aligned}$$

where $q_i(0)$ is the i^{th} column of $Q(0)$. The values are in metres for the first two rows, and in radians for the third. The communication radii of the have values $R_1 = 0.83m$, $R_2 = 0.90m$, $R_3 = 0.31m$, and $R_4 = 0.70m$. The parameter ψ_i^{max} is equal to $1rad$ for all the robots. The task velocities are given by

$$\begin{aligned} u_1^e &= [-3m/s \quad -3m/s \quad 0rad/s]^T \\ u_2^e &= [3m/s \quad 3m/s \quad 0rad/s]^T \\ u_3^e &= [3\sin(t/2)m/s \quad 3\cos(t/2)m/s \quad 0rad/s]^T \\ u_4^e &= [0m/s \quad 0m/s \quad 0rad/s]^T \end{aligned}$$

The task velocities u_i^e are such that if the robots were to move according to u_i^e alone, the sensing network would eventually lose strong connectivity. In Figure 1 we see that the initial condition is such that the sensing network is strongly connected. The field-of-view is indicated by the sector with an acute angle. Under the action of u_i^c given by (13), the sensing network remains strongly connected as seen in Figure 2. These positions are such that the agents form a network which is barely strongly connected, as seen from Figure 2 where the

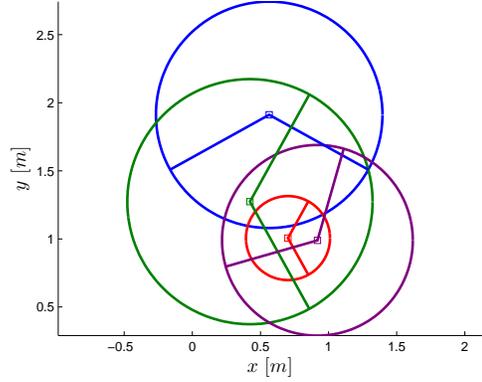


Fig. 1. Positions of the four robots at $t = 0$, denoted by squares. The boundary of the circular communication region of each robot is indicated by the circle. The field-of-view of each robot is denoted by the smaller sector of each circle.

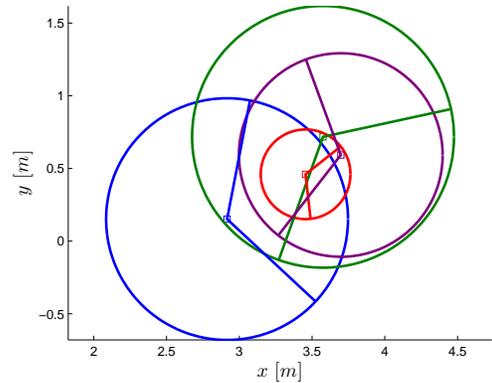


Fig. 2. Final positions and orientations of the four robots. The sensing graph remains strongly connected.

agents are located close to the boundaries of the field-of-view of other robots. The evolution of the state $q(t)$ is shown in Figure 3.

VI. CONCLUSION

In this paper, we have proposed a control method to preserve the strong connectivity of the directed sensing graph formed by robots with limited field-of-view sensors. The proposed control is shown to guarantee preservation of strong connectivity in the directed sensing network even in the presence of additional disturbance terms, without favoring any particular graph topology. Note that the terms u_i^e can also be interpreted as additional control objectives such as tracking a leader or reaching positions in a formation. Therefore, the connectivity preserving control law presented in this paper can be used alongside other control tasks with higher priority.

In order to implement such a control law, robots may have to be able to find the direction of the transmitter from which it receives information. This can be done using radio direction finding techniques. Of course, this requirement adds to the complexity required of the robots that can implement such a control method. For some applications in which strong con-

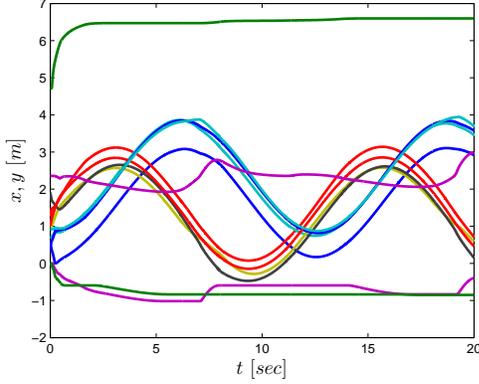


Fig. 3. Trajectories $x_i(t)$, $y_i(t)$, and $\theta_i(t)$ for the four agents.

nectivity of the sensing network (and therefore communication network) at all times is critical, this may be justified.

Future work consists of analyzing the effects of the time delay in estimating the quantities $\lambda_2(\mathcal{L})$ and γ . Since these quantities are only used in order to choose which edges must be preserved, the effects of small delays are not expected to be serious. If the speed of motion of the robots is much faster than the time constant related to the estimation of these dynamic quantities, then strong connectivity may be lost. This motivates techniques which may limit the motion of the robots depending on the time taken for information related to changes in network topology to propagate through the network.

REFERENCES

- [1] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, Jan 2007.
- [2] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, May 2005.
- [3] M. Zavlanos, M. Egerstedt, and G. Pappas, "Graph-theoretic connectivity control of mobile robot networks," *Proceedings of the IEEE*, vol. 99, no. 9, pp. 1525–1540, sept. 2011.
- [4] M. Zavlanos and G. Pappas, "Distributed connectivity control of mobile networks," *Robotics, IEEE Transactions on*, vol. 24, no. 6, pp. 1416–1428, dec. 2008.
- [5] P. Yang, R. A. Freeman, G. J. Gordon, K. M. Lynch, S. S. Srinivasa, and R. Sukthankar, "Decentralized estimation and control of graph connectivity for mobile sensor networks," *Automatica*, vol. 46, no. 2, pp. 390–396, 2010.
- [6] L. Sabattini, N. Chopra, and C. Secchi, "Distributed control of multi-robot systems with global connectivity maintenance," in *Intelligent Robots and Systems (IROS), 2011 IEEE/RSJ International Conference on*, Sept 2011, pp. 2321–2326.
- [7] —, "Decentralized connectivity maintenance for cooperative control of mobile robotic systems," vol. 32, no. 12, pp. 1411–1423, October 2013.
- [8] P. Robuffo Giordano, A. Franchi, C. Secchi, and H. H. Bulthoff, "A passivity-based decentralized strategy for generalized connectivity maintenance," *The International Journal of Robotics Research*, vol. 32, no. 3, pp. 299–323, 2013.
- [9] L. Sabattini, C. Secchi, and N. Chopra, "Decentralized control for maintenance of strong connectivity for directed graphs," in *Control Automation (MED), 2013 21st Mediterranean Conference on*, June 2013, pp. 978–986.
- [10] M. M. Asadi and A. G. Aghdam, "Potential-based flocking in multi-agent systems with limited angular fields of view," in *2014 American Control Conference*, June 2014, pp. 299–304.

- [11] D. D. Paola, R. D. Asmundis, A. Gasparri, and A. Rizzo, "Decentralized topology control for robotic networks with limited field of view sensors," in *2012 American Control Conference (ACC)*, June 2012, pp. 3167–3172.
- [12] H. A. Poonawala, "Formation control and connectivity control for mobile robot networks using vision based measurements," *PhD Thesis*, 2014.
- [13] H. Poonawala and M. Spong, "Preserving strong connectivity in directed proximity graphs," *Automatic Control, IEEE Transactions on*, In review. [Online]. Available: <http://users.ices.utexas.edu/~hap/HPOonawalaTAC.pdf>
- [14] C. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems*. World Scientific Publishing Company, Incorporated, 2007. [Online]. Available: <http://books.google.com/books?id=vMUexcIVnjIC>
- [15] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 2, pp. 298–305, 1973.
- [16] H. Minc, *Nonnegative matrices*. Technion-Israel Institute of Technology, Dept. of Mathematics, 1974.
- [17] C. W. Wu, "Algebraic connectivity of directed graphs," *Linear and Multilinear Algebra*, vol. 53, no. 3, pp. 203–223, 2005.
- [18] Z. Qu, C. Li, and F. Lewis, "Cooperative control with distributed gain adaptation and connectivity estimation for directed networks," *International Journal of Robust and Nonlinear Control*, pp. n/a–n/a, 2012. [Online]. Available: <http://dx.doi.org/10.1002/rnc.2895>
- [19] H. Poonawala and M. Spong, "Decentralized estimation of the algebraic connectivity for strongly connected networks," in *American Control Conference (ACC), 2015*, July 2015, pp. 4068–4073.
- [20] M. Ji and M. Egerstedt, "Distributed coordination control of multiagent systems while preserving connectedness," *Robotics, IEEE Transactions on*, vol. 23, no. 4, pp. 693–703, aug. 2007.

APPENDIX

PROOF OF THEOREM IV.1

We define edge potential functions as

$$\bar{V}_{ij}^d(q) = \int_0^{d_{ij}} s \alpha_{ij}(s) ds \quad (20)$$

and

$$\bar{V}_{ij}^\psi(q) = \int_0^{|\psi_{ij}|} s \beta_{ij}(s) ds, \quad (21)$$

where we have suppressed the dependence of α_{ij} and β_{ij} on various parameters. These edge potentials can be summed over all edges to obtain the overall potential function

$$\bar{V}(q) = \sum_{(i,j) \in E_c} \bar{V}_{ij}^d(q) + \bar{V}_{ij}^\psi(q). \quad (22)$$

The function $\bar{V}(q)$ serves as a Lyapunov-like function. If $d_{ij} \rightarrow R_{ij}$ or $|\psi_{ij}| \rightarrow \psi_i^{max}$ for edge $(i, j) \in E_c$, then $\bar{V}(q) \rightarrow \infty$. The proof will rest on showing that $\bar{V}(q)$ remains finite for all time.

We first prove the following result.

Lemma A.1. *Let \mathcal{L}_σ be a symmetric graph Laplacian for a connected undirected graph $\vec{G} = (V, \vec{E})$ with edge weights σ_{ij} . Let $|V| = p$, and $x \in \mathbb{R}^p$. Let $q \in \mathbb{N}$, and $z \in \mathbb{R}^p$ be any vector. Consider the quantity Ω given by*

$$\Omega = \left\| \begin{bmatrix} \mathcal{L}_\sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \right\| \quad (23)$$

If $x \neq c\mathbf{1}_p$ for any $c \in \mathbb{R}$, then $\Omega > 0$. Moreover, if $\sigma_{ij} \rightarrow \infty \forall (i, j) \in \vec{E}$, then $\Omega \rightarrow \infty$.

Proof: Let $P \in \mathbb{R}^{p \times p-1}$ be any matrix such that $P^T \mathbf{1}_p = 0$ and $P^T P = I_{p-1}$, where I_{p-1} is the identity matrix of size

$p - 1$. Let $x \in R^p$ be any vector. We can define $x^P = P^T x$, and thus express x as

$$x = Px^P + \left(\frac{x^T \mathbf{1}_p}{p} \right) \mathbf{1}_p \quad (24)$$

One can check that $Px_P = 0 \iff x = c\mathbf{1}_p$, where c is a real number. Given a symmetric graph Laplacian $\mathcal{L}_\sigma \in \mathbb{R}^{p \times p}$ and the vector x , we have that

$$\|\mathcal{L}_\sigma x\| \geq \lambda_2(\mathcal{L}_\sigma) \|Px^P\| \quad (25)$$

If \overleftrightarrow{G} is connected then $\lambda_2(\mathcal{L}_\sigma) > 0$. If $x \neq c\mathbf{1}_p$ for any $c \in \mathbb{R}$, then $Px_P \neq 0$ and therefore $\|\mathcal{L}_\sigma x\| > 0$. For any undirected graph \overleftrightarrow{G} , $\lambda_2(\mathcal{L}_\sigma)$ is a non-decreasing function of the edge weights [5]. Consider the weighted symmetric graph Laplacian $\mathcal{L}_{\sigma, \min}$ where every edge weight σ_{ij} is replaced by $\sigma_{\min} = \min_{(i,j) \in \overleftrightarrow{E}} \sigma_{ij}$. Now, this matrix is converted to \mathcal{L}_σ by increasing each edge weight. This means that

$$\lambda_2(\mathcal{L}_\sigma) \geq \lambda_2(\mathcal{L}_{\sigma, \min}) \quad (26)$$

However,

$$\begin{aligned} \lambda_2(\mathcal{L}_{\sigma, \min}) &= \lambda_2(\sigma_{\min} \mathcal{L}_1) \\ &= \sigma_{\min} \lambda_2(\mathcal{L}_1) \end{aligned} \quad (27)$$

where \mathcal{L}_1 is the graph Laplacian of \overleftrightarrow{G} obtained when edge weights are either zero or one. Thus, we can conclude that

$$\lambda_2(\mathcal{L}_\sigma) \geq \sigma_{\min} \lambda_2(\mathcal{L}_1) \quad (28)$$

where $\lambda_2(\mathcal{L}_1) > 0$ since \overleftrightarrow{G} is connected. We have assumed that for all edges $(i, j) \in \overleftrightarrow{E}$ we have that $\sigma_{ij}(t) \rightarrow \infty$ as $t \rightarrow T$. Since there are a finite number of edges, if $\sigma_{ij} \rightarrow \infty \forall (i, j) \in \overleftrightarrow{E}$, then it holds that $\sigma_{\min} \rightarrow \infty$.

Clearly, $\Omega = \|\mathcal{L}_\sigma x\| > \sigma_{\min} \lambda_2(\mathcal{L}_1) \|Px^P\|$. Since \overleftrightarrow{G} is connected and $x \neq c\mathbf{1}_p$ for any $c \in \mathbb{R}$, it must hold that $\Omega \rightarrow \infty$ as $\sigma_{\min} \rightarrow \infty$. ■

We now rewrite the equations of the robots in such a way that the k^{th} elements of all vectors q_i are combined. This was also done in [20]. We will sometimes denote the k^{th} element of a vector x as $[x]_k$, in order to avoid confusion in subscripts.

We can rewrite dynamics (3) as

$$\dot{\mathbf{x}} = \mathbf{u}_x^c + \mathbf{u}_x^e \quad (29)$$

$$\dot{\mathbf{y}} = \mathbf{u}_y^c + \mathbf{u}_y^e \quad (30)$$

$$\dot{\Theta} = \mathbf{u}_\theta^c + \mathbf{u}_\theta^e \quad (31)$$

where $[\mathbf{x}]_k = x_k$, $[\mathbf{y}]_k = y_k$, $[\Theta]_k = \theta_k$, $[\mathbf{u}_x^c]_i = [u_x^c]_1$, $[\mathbf{u}_y^c]_i = [u_y^c]_2$, $[\mathbf{u}_\theta^c]_i = [u_\theta^c]_3$, $[\mathbf{u}_x^e]_i = [u_x^e]_1$, $[\mathbf{u}_y^e]_i = [u_y^e]_2$, $[\mathbf{u}_\theta^e]_i = [u_\theta^e]_3$.

We have that

$$\frac{\partial d_{ij}}{\partial q_i} = [(x_i - x_j)/d_{ij}^2 \quad (y_i - y_j)/d_{ij}^2 \quad 0] \quad (32)$$

and

$$\frac{\partial |\psi_{ij}|}{\partial q_i} = \text{sign}(\psi_{ij}) [(y_i - y_j)/d_{ij}^2 \quad (x_j - x_i)/d_{ij}^2 \quad -1]. \quad (33)$$

Consider the set D^* given by

$$D^* = \{q \in \mathbb{R}^{3N} : d_{ij} = R_{ij} \text{ or } |\psi_{ij}| = \psi_i^{\max} \forall (i, j) \in E_c\}. \quad (34)$$

The set D^* is a subset of the boundary of set D in (19). If $q(t)$ reaches D^* at some time T , this means that all edges in the graph are lost at the same time instant. We first show that this cannot happen.

Lemma A.2. Consider the fixed directed graph $G_c = (V, E_c)$ with dynamics (3). Let the control u_i^c be selected according to (13). The external control $u_i^e(t)$ is unknown but bounded for each $i \in V$. Then, for any solution $q(t)$ of (3) with initial condition $q(t_0) \in D$, if $q(t) \in D$ for all $t \in [t_0, T)$ where $T > t_0$, then $q(T) \notin D^*$.

Proof: We prove the Lemma by contradiction. Assume that at $t = T$, $q(T) \in D^*$. This means that for all $(i, j) \in E_c$, $\lim_{t \rightarrow T} \alpha_{ij}(d_{ij}(t)) = \infty$ or $\lim_{t \rightarrow T} \beta_{ij}(|\psi_{ij}(t)|) + \beta_{ji}(|\psi_{ji}(t)|) = \infty$. Consider an undirected graph $\overleftrightarrow{G}_c = (V, \overleftrightarrow{E}_c)$, where edge set \overleftrightarrow{E}_c is such that $(i, j) \in \overleftrightarrow{E}_c$ iff (i, j) or $(j, i) \in E_c$.

Consider the weighted **symmetric** Laplacian \mathcal{L}_α with edge weights of the form $\alpha_{ij} + \alpha_{ji}$ corresponding to a graph $G' = (V, E')$, where $E' \subseteq \overleftrightarrow{E}_c$ and $(i, j) \in E'$ if and only if $\lim_{t \rightarrow T} \alpha_{ij}(d_{ij}(t)) = \infty$. Similarly, we construct a symmetric Laplacian \mathcal{L}_β with edge weights of the form $\beta_{ij} + \beta_{ji}$ if and only if $\lim_{t \rightarrow T} \beta_{ij}(|\psi_{ij}(t)|) + \beta_{ji}(|\psi_{ji}(t)|) = \infty$.

We can then rewrite the control terms \mathbf{u}_x^c , \mathbf{u}_y^c , and \mathbf{u}_θ^c using (32) and (33) in (13), to obtain

$$\mathbf{u}_x^c = -\mathcal{L}_\alpha \mathbf{x} - \mathcal{L}_\beta \mathbf{y} + \mathbf{z}_1, \quad (35)$$

$$\mathbf{u}_y^c = -\mathcal{L}_\alpha \mathbf{y} + \mathcal{L}_\beta \mathbf{x} + \mathbf{z}_2, \text{ and} \quad (36)$$

$$[\mathbf{u}_\theta^c]_i = \sum_{\{j:(i,j) \in E_c\}} -\psi_{ij} \beta_{ij}(|\psi_{ij}|) \text{sign}(\psi_{ij}). \quad (37)$$

where the vectors \mathbf{z}_1 and \mathbf{z}_2 absorb those terms from the control (13) for which $\lim_{t \rightarrow T} \alpha_{ij}(d_{ij}(t)) < \infty$ and $\lim_{t \rightarrow T} \beta_{ij}(|\psi_{ij}(t)|) + \beta_{ji}(|\psi_{ji}(t)|) < \infty$. We can treat these vectors as bounded disturbance terms when analyzing the control.

Since $\frac{d}{dx} \int_0^y sf(s)ds = yf(y) \frac{dy}{dx}$, it can also be shown that $(\mathbf{u}_x^c)^T = -\frac{\partial \bar{V}}{\partial \mathbf{x}}$, $\mathbf{u}_y^c = \frac{\partial \bar{V}}{\partial \mathbf{y}}$, and $[\mathbf{u}_\theta^c]_i = \left[\frac{\partial \bar{V}}{\partial \Theta} \right]_i$.

The derivative of $\bar{V}(q)$ is

$$\begin{aligned} \dot{\bar{V}}(q) &= \frac{\partial \bar{V}}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial \bar{V}}{\partial \mathbf{y}} \dot{\mathbf{y}} + \frac{\partial \bar{V}}{\partial \Theta} \dot{\Theta} \\ &= (\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y} + \mathbf{z}_1)^T (-\mathcal{L}_\alpha \mathbf{x} - \mathcal{L}_\beta \mathbf{y} - \mathbf{z}_1 + \mathbf{u}_x^e) \\ &\quad + (\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x} + \mathbf{z}_2)^T (-\mathcal{L}_\alpha \mathbf{y} + \mathcal{L}_\beta \mathbf{x} - \mathbf{z}_2 + \mathbf{u}_y^e) \\ &\quad + \sum_{i=1}^N [\mathbf{u}_\theta^c]_i (-[\mathbf{u}_\theta^c]_i + [\mathbf{u}_\theta^e]_i) \\ &= -\|\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y}\|^2 + (\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y})^T (\mathbf{u}_x^e - 2\mathbf{z}_1) \\ &\quad - \|\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x}\|^2 + (\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x})^T (\mathbf{u}_y^e - 2\mathbf{z}_2) \dots \\ &= -\|\mathbf{u}_\theta^c\|^2 + (\mathbf{u}_\theta^e)^T \mathbf{u}_\theta^e + \Delta_1 \\ &= -\|\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y}\|^2 - \|\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x}\|^2 - \|\mathbf{u}_\theta^c\|^2 + \Delta_1 + \Delta_2 \end{aligned} \quad (38)$$

where

$$\Delta_1 = \mathbf{z}_1^T (\mathbf{u}_x^e - \mathbf{z}_1) + \mathbf{z}_2^T (\mathbf{u}_y^e - \mathbf{z}_2) \quad (39)$$

and

$$\begin{aligned} \Delta_2 &= (\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y})^T (\mathbf{u}_x^e - 2\mathbf{z}_1) \\ &\quad + (\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x})^T (\mathbf{u}_y^e - 2\mathbf{z}_2) + (\mathbf{u}_\theta^c)^T \mathbf{u}_\theta^e \\ &\leq (\|\mathcal{L}_\alpha \mathbf{x} \\ &\quad + \mathcal{L}_\beta \mathbf{y}\|) \|\mathbf{u}_x^e - 2\mathbf{z}_1\| + (\|\mathcal{L}_\alpha \mathbf{y} + \mathcal{L}_\beta \mathbf{x}\|) \|\mathbf{u}_y^e - 2\mathbf{z}_2\| + \|\mathbf{u}_\theta^c\| \|\mathbf{u}_\theta^e\| \end{aligned} \quad (40)$$

We can bound $\dot{\bar{V}}(q)$ as

$$\begin{aligned} \dot{\bar{V}}(q) &\leq -\|\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y}\| (\|\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y}\| - \|\mathbf{u}_x^e - 2\mathbf{z}_1\|) \dots \\ &\quad - \|\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x}\| (\|\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x}\| - \|\mathbf{u}_y^e - 2\mathbf{z}_2\|) \dots \\ &\quad - \|\mathbf{u}_\theta^c\| (\|\mathbf{u}_\theta^c\| - \|\mathbf{u}_\theta^e\|) + \|\Delta_1\| \end{aligned} \quad (41)$$

The terms \mathbf{u}_x^e , \mathbf{u}_y^e , \mathbf{u}_θ^e , \mathbf{z}_1 and \mathbf{z}_2 are all bounded by assumption. Let the norm of all terms in (41) consisting of only these terms be bounded above by M . When the norm of $\|\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y}\|$, $\|\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x}\|$, or $\|\mathbf{u}_\theta^c\|$ is larger than M , then the corresponding term in (41) is negative. This implies that we can bound the values of Δ_2 for which $\dot{\bar{V}}(q) > 0$ from above by $3M^2$. Thus, if one of the negative definite terms in (38) has magnitude greater than $3M^2 + M$, then $\dot{\bar{V}}(q) < 0$, where the additional M comes from the term Δ_1 .

By applying Lemma A.1, we can show that one of the terms out of $\|\mathcal{L}_\alpha \mathbf{x}\|$, $\|\mathcal{L}_\alpha \mathbf{y}\|$, $\|\mathcal{L}_\beta \mathbf{x}\|$, and $\|\mathcal{L}_\beta \mathbf{y}\|$ must approach infinity as $t \rightarrow T$. This implies that one of the terms out of $\|\mathcal{L}_\alpha \mathbf{x} + \mathcal{L}_\beta \mathbf{y}\|$ and $\|\mathcal{L}_\alpha \mathbf{y} - \mathcal{L}_\beta \mathbf{x}\|$ must become unbounded as $t \rightarrow T$. This in turn implies that there exists a time τ where $0 < \tau < T$ such that $\dot{\bar{V}}(q(t)) < 0$ for all $t \in (\tau, T]$. Since $\dot{\bar{V}}(q(t)) < 0$ for all $t \in (\tau, T]$, we have that $\bar{V}(q(T))$ remains finite, which contradicts the assumption that $q(T) \in D^*$, since for all $q^* \in D^*$, $\bar{V}(q^*) = \infty$. This proves the Lemma. ■

A. Theorem IV.1

Proof: We again prove the result by the method of contradiction. Let there exist some $T > t_0$ such that the solution $q(t)$ of (3) for initial condition $q(t_0)$ is such that $q(t) \in D \forall t \in [t_0, T)$ and $q(T) \notin D$. Let some set of edges E^* be broken at $t = T$. The edges E^* may consist of multiple disconnected components. Each of these disconnected components satisfy the conditions of Theorem A.2 independently, since the control terms due to the edges between these components are not broken at $t = T$, and can be considered as bounded disturbance terms. Therefore we obtain a contradiction, and so conclude that $q(t) \in D \forall t \geq t_0$. ■