Stability Analysis Via Refinement Of Piece-wise Linear Lyapunov Functions

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Abstract—We present an algorithm for finding piece-wise linear Lyapunov functions that verify the asymptotic stability of piece-wise affine differential inclusions, where the latter can over-approximate a large class of nonlinear or discontinuous dynamical systems. Existing methods either use a fixed set of pieces (a partition) to define the Lyapunov function, or use heuristic methods to split the pieces, thereby refining the partition. Our algorithm involves iteratively refining partitions using an exact criterion. This criterion enables us to show that the algorithm is sound and complete.

I. INTRODUCTION

Lyapunov’s second method is the method most commonly used to verify the stability properties of nonlinear dynamical systems. As is well known, this method involves finding a positive definite function whose value decreases along solutions of the dynamical system. There is no general method to find this function, called the Lyapunov function, for all dynamical systems.

Often, one chooses to develop a parametrized structure for the Lyapunov function given a parametrized structure of the dynamical system. The value of this parametrization is that one can simultaneously propose an algorithm that computes the parameters of the Lyapunov function given the parameters of the dynamical system. Essentially, the structure of the system is exploited when searching for a valid Lyapunov function.

This paper focuses on searching for piece-wise linear Lyapunov functions to analyze the properties of piece-wise affine differential inclusions. Blanchini [1] first proposed using piece-wise linear Lyapunov functions to analyze dynamical systems. Later, Johansson proposed conditions [2] that verify whether a candidate Lyapunov function is a valid Lyapunov function. The parameters of this Lyapunov function include the partition (pieces) over which the Lyapunov function is defined, and the linear function associated with each piece in the partition. The conditions in [2] facilitate an efficient search for the linear functions associated with the pieces in a fixed partition. This search takes the form of a linear program.

Johansson suggests [2] refining pieces by partitioning them as one way to change the Lyapunov function when it fails to satisfy the stability condition. The main idea he proposes is to solve the dual problem to the linear program and to then use the optimal dual variables to determine which cell should be refined. The refinement step involves splitting a cell by splitting a boundary. The properties of this heuristic approach are not investigated. This approach is a form of counter-example guided refinement (CEGAR). Other authors propose CEGAR-based search methods for piece-wise constant dynamics [3] or piece-wise quadratic Lyapunov functions [4].

Contribution: Our main contribution is to propose a method to partition the pieces defining a piece-wise linear Lyapunov function that leads to a sequential optimization-based algorithm for searching for such functions that verify stability properties of piece-wise affine differential inclusions. Our refinement step enables precise analysis of this sequential optimization approach; we show that the algorithm is sound and complete. An advantage of our approach over similar methods is that piece-wise linear Lyapunov functions are a universal class of Lyapunov functions [1], and piece-wise affine differential inclusions can over-approximate a large class of nonlinear systems relatively efficiently.

Overview: Sections II and III introduce piece-wise dynamical systems and Lyapunov functions and the conditions on their parameters that lead to verification of the system’s stability properties. Section IV describes the refinement procedure we propose and the resulting algorithm for finding piece-wise linear Lyapunov functions for verifying piece-wise affine differential inclusions. Section V demonstrates the utility of our algorithm by applying it to representative dynamical systems.
II. Preliminaries

**Notation:** The indices of the elements of a finite set $S$ form the set $I_S$. We denote the convex hull of a set $S$ by $\text{conv}(S)$, and the interior of $S$ by $\text{Int}(S)$. $\text{pos}(S)$ and $\text{pos}_{>0}(S)$ respectively denote the set of non-negative and strictly positive combinations of the elements of a set $S$.

The vector $1_n \in \mathbb{R}^n$ has all elements equal to unity. We omit the subscript $n$ if its value is clear from the context. The set $\text{GL}(n)$ represents the set of $n \times n$ matrices with non-zero determinant. The rank of a matrix $E \in \mathbb{R}^{m \times n}$ is given by $r(E)$.

A. Partitions And Refinements

A partition $\mathcal{P}$ is a collection of subsets $\{X_i\}_{i \in I_P}$, where $I_P$ is an index set, $X_i \subseteq \mathbb{R}^n$ for each $i \in I_P$, $n \in \mathbb{N}$, and $\text{Int}(X_i) \cap \text{Int}(X_j) = \emptyset$ for each pair $i, j \in I_P$ such that $i \neq j$. We refer to $\cup_{i \in I_P} X_i$ as the domain of $\mathcal{P}$, which we also denote by $\text{Dom}(\mathcal{P})$. We also refer to the subsets $X_i$ in $\mathcal{P}$ as the cells of the partition. Note that our definition of a partition allows some cells in $\mathcal{P}$ to be the boundary of other cells in $\mathcal{P}$, which is useful for handling sliding modes.

Let $\mathcal{P} = \{Y_i\}_{i \in J_P}$ and $\mathcal{R} = \{Z_j\}_{j \in J_R}$ be two partitions of a set $S = \text{Dom}(\mathcal{P}) = \text{Dom}(\mathcal{R})$. A partition $\mathcal{R}$ is a refinement of $\mathcal{P}$ if $Z_j \cap Y_i \neq \emptyset$ implies that $Z_j \subseteq Y_i$. We denote the set of refinements of a partition $\mathcal{P}$ as $\text{Ref}(\mathcal{P})$. There exists a natural abstraction function $\pi_{\mathcal{R} \rightarrow \mathcal{P}} : I_R \mapsto I_P$, given by $\pi_{\mathcal{R} \rightarrow \mathcal{P}}(j) = \{i \in I_P : Z_j \subseteq Y_i\}$.

B. Polyhedral Cones

We consider partitions $\mathcal{P}$ where each cell $X_i \in \mathcal{P}$ is a polyhedral cone with apex at the origin. Each cell $X_i \in \mathcal{P}$ is of the form $X_i = \{x \in \mathbb{R}^n : Ex \geq 0\}$, where $E \in \mathbb{R}^{m \times n}$ and its rank $r(E) = n$. The $i^{th}$ row $E^i$ of matrix $E$ defines a hyperplane $\{x \in \mathbb{R}^n : E^i x = 0\}$.

The dual cone $Z^*$ to a convex set $Z$ is the set $\{y \in (\mathbb{R}^n)^* : \langle y, x \rangle \geq 0 \forall x \in Z\}$, where $(\cdot, \cdot): Z^* \times Z \rightarrow \mathbb{R}$ is the evaluation of $x \in \mathbb{R}^n$ by the functional $y \in (\mathbb{R}^n)^*$. This cone lies in the dual space to $\mathbb{R}^n$, however since $\mathbb{R}^n$ is self-dual, we can identify the dual cone with a cone in $\mathbb{R}^n$. Some authors refer to this second cone as the internal dual cone.

If a polyhedral cone $X$ is given by

$$X = \{x \in \mathbb{R}^n : Ex \geq 0\},$$

its dual cone $X^*$ is given by

$$X^* = \left\{ y \in (\mathbb{R}^n)^* : y = \sum_{i \in \{1, \ldots, n\}} E^i \mu_i, \mu_i \geq 0 \right\},$$

and its interior $\text{Int}(X^*)$ is given by

$$\text{Int}(X^*) = \text{pos}_{>0} \left( \{ E^i \} \right) = \left\{ y \in (\mathbb{R}^n)^* : y = \sum_{i \in \{1, \ldots, n\}} E^i \mu_i, \mu_i > 0 \right\}.$$  

The polar cone $X^o$ of $X$ is simply $-X^*$. Since we consider cones in $\mathbb{R}^n$, these cones defined in the dual space have corresponding cones in $\mathbb{R}^n$.

C. Piece-wise Affine Differential Inclusions

A piece-wise affine differential inclusion $\Omega_{\mathcal{P}}$ associated with partition $\mathcal{P} = \{X_i\}_{i \in I_P}$ is a collection,

$$\Omega_{\mathcal{P}} = \{ \mathcal{A}_i(x) \}_{i \in I_P}$$

that to each cell $X_i \in \mathcal{P}$ assigns the affine differential inclusion $\mathcal{A}_i(x) = \text{conv} \left( \{ A_{ij} x + a_{ij} \}_{j \in I_{A_i}} \right)$. Therefore,

$$\dot{x}(t) \in \mathcal{A}_i(x), \text{ if } x_i(t) \in X_i.$$  

The cell $X_i \in \mathcal{P}$ is given by

$$X_i = \{ x \in \mathbb{R}^n : F_i x \geq 0 \}.$$  

We assume that $F_i$ is full rank for each $i \in I_P$. Furthermore, we assume that $0 \in \text{Int} (\text{Dom}(\mathcal{P}))$. This assumption ensures that $X_i$ is a polyhedral cone with apex at the origin and non-empty interior. Figure 1 depicts an example of such a piece-wise affine differential inclusion where $|I_P| = 4$.

D. Piece-wise Linear Lyapunov Functions

Piece-wise linear (PWL) Lyapunov functions were first studied in [1]. Since then, computational methods to find such functions have also been introduced [2], [5]. If the level sets of the PWL Lyapunov function are convex polyhedra, then the Lyapunov function is also known as a polytopic function. Such functions can be represented as the $\infty$-norm of a linear function of the state.

The advantage of this class of Lyapunov functions are two-fold. First, they are shown to be a universal subset of Lyapunov functions [1], unlike quadratic functions. Secondly, given a partition that defines the pieces of the function, the search for a PWL Lyapunov function boils down to a linear program [1], [2].

We parameterize a continuous polyhedral Lyapunov function $V_Q(x)$ with a partition $Q = \{Z_i\}_{i \in I_Q}$ and a collection of
vectors \( \{ p_i \}_{i \in I_\ell} \) such that \( V_Q(x) = p_i^T x \), if \( x \in Z_i \subseteq \mathbb{R}^n \).

Each set \( Z_i \in \mathcal{Q} \) is given by \( Z_i = \{ x \in \mathbb{R}^n : E_i x \geq 0 \} \), and we assume that \( r(E_i) = n \). Figure 2 shows an example of such a partition, and level sets due to two different sets \( \{ p_i \}_{i \in I_\ell} \) associated with this partition. This definition of \( V_Q(x) \) corresponds to polytopic Lyapunov functions [1], [2], which is a subclass of piece-wise linear Lyapunov functions.

### III. Stability Conditions

Our concern is with the stability properties of the origin \( x = 0 \) of a piece-wise affine differential inclusion \( \Omega_P \) in (4). To establish these properties, we will use a PWL Lyapunov function, also known as a polytopic Lyapunov function.

We assume that the piece-wise affine dynamical system satisfies the following assumption [6]:

\[
A1 \quad 0 \in \text{conv}\left( \{ a_{ij} \}_{i \in I_P, j \in I_{A_i}} \right).
\]

This assumption ensures that the origin is at least a switched equilibrium when some of the vector fields \( A_i \) are non-zero at the origin. When \( a_{ij} = 0 \) for every \( i \in I_P, j \in I_{A_i} \), then the origin is an equilibrium point.

To verify the stability properties of \( \Omega_P \) in (4), we use a Lyapunov function \( V_Q(x) \) with partition \( \mathcal{Q} = \{ Z_i \}_{i \in I_\ell} \) and parameters \( \{ p_i \}_{i \in I_\ell} \). Partition \( \mathcal{Q} \) is a refinement of \( \mathcal{P} \), i.e., \( \mathcal{Q} \in \text{Ref}(\mathcal{P}) \) (see Section II-A). The abstraction function \( \pi_{Q \rightarrow P} \) associates each cell in \( \mathcal{Q} \) with a unique cell in \( \mathcal{P} \). Most methods for finding piece-wise Lyapunov function choose \( \mathcal{Q} = \mathcal{P} \), so that \( \pi_{Q \rightarrow P} \) is the identity map.

Sufficient conditions on a piece-wise differential inclusion and candidate Lyapunov function that certify the existence of the control properties under consideration are given in [7]. We now introduce some sets and variables related to the piece-wise nature of the problem. Let \( I_{dec}(Q, P) = I_Q \times I_{A_k} \) where \( k = \pi_{Q \rightarrow P}(i) \). Let \( I_{cont}(Q) \subseteq I_Q \times I_Q \) be the set of pairs of indices \( (i, j) \) such that \( Z_i \cap Z_j \neq \emptyset \), which are cells with a common boundary. This boundary is given by

\[
Z_i \cap Z_j = \{ x \in \mathbb{R}^n : n_{ij}^T x = 0 \}.
\]

The subscript \( \text{dec and cont} \) are abbreviations of ‘decrease’ and ‘continuous’ respectively.

We define variables \( \alpha_{ik}, \beta_{ik} \) for each \( j \in I_{A_i} \) where \( i \in I_P \). Let \( \alpha_{ij} = 1 \) if \( A_j \neq 0 \) and \( \alpha_{ij} = 0 \) otherwise. Similarly, let \( \beta_{ij} = 1 \) if \( a_{ij} \neq 0 \) and \( \alpha_{ij} = 0 \) otherwise.

The conditions for asymptotic stability of the origin of the piece-wise affine differential inclusion is given in the following result, adapted from [8], [2].

**Lemma 1.** Let \( \Omega_P \) be a piece-wise affine differential inclusion as in (4) satisfying assumption A1 and \( V_Q \) be a candidate PWL Lyapunov function with partition \( \mathcal{Q} = \{ Z_i \}_{i \in I_\ell} \), parameters \( \{ p_i \}_{i \in I_\ell} \), and abstraction function \( \pi_{Q \rightarrow P} \). Each set \( Z_i \) is \( \{ x \in \mathbb{R}^n : E_i x \geq 0 \} \). Let \( I_Q \), \( I_{dec}(Q, P) \) and \( I_{cont}(Q) \) be the index sets associated with \( \Omega_P \) and \( V_Q \). If the set of constraints

\[
\begin{align*}
 p_i &= E_i^T \mu_i, \quad \forall i \in I_Q, \\
 \mu_i &\geq 1, \quad \forall i \in I_Q, \\
 \begin{bmatrix}
 E_i^T & 0 \\
 0 & 1
\end{bmatrix} \nu_{ij} &\geq -\begin{bmatrix} A_{\pi_{Q \rightarrow P}(ij)} \end{bmatrix} p_i \\
 &\quad \forall (i, j) \in I_{dec}(Q, P), \\
 \nu_{ij} &\geq \begin{bmatrix} \alpha_{\pi_{Q \rightarrow P}(ij)} \beta_{\pi_{Q \rightarrow P}(ij)} \end{bmatrix} \frac{1}{1}, \quad \forall (i, j) \in I_{dec}(Q, P), \\
 p_i - p_j &= \lambda_{ij} \eta_{ij}, \quad \forall (i, j) \in I_{cont}(Q)
\end{align*}
\]

is feasible then the origin of \( \Omega_P \) is asymptotically stable.

**Proof.** See [8] for the full proof. These conditions are due to a variant of Farkas’ Lemma [9], recounted in Lemma 2 below. Briefly, conditions (8) and (9) constrain \( V_P(x) \) to be positive definite, condition (10) constrains \( V_P(x) \) to be continuous, and conditions (11) and (12) require \( V_P(x) \) to decrease along solutions of \( \Omega_P \). The variables \( \mu_i, \nu_{ij} \) and \( \lambda_{ij} \) serve as certificates for positivity or negativity of the related function.

**Remark 1 (Sliding Modes).** The Lyapunov function \( V_Q(x) \) and conditions in (8)-(12) do not include conditions for the case where sliding occurs at the boundaries of cells in \( \mathcal{P} \). One may define a boundary where sliding occurs using constraints of the form \( Ex \geq 0 \), and then define a differential inclusion to represent the sliding motion, and a linear function as a Lyapunov function for the boundary. This step would lead to conditions similar to (8)-(12) and conclusions similar to that of Lemma 1 but with a more tedious notation. We therefore avoid explicitly including sliding modes in (8)-(12).

**Lemma 2** ([2], [9]). Let \( Z = \{ x \in \mathbb{R}^n : Ex \geq 0 \} \) where \( E \in \mathbb{R}^{m \times n} \) has rank \( r(E) = n \) and \( Z \setminus \{0\} \) is non-empty. Given \( v \in \mathbb{R}^n \), the following are equivalent

\[
\begin{align*}
 i) \quad & v \in \text{Int}(Z^*) \\
 ii) \quad & Ex \geq 0, x \neq 0 \implies v^T x > 0 \\
 iii) \quad & \exists \mu \in \mathbb{R}^n \text{ such that } \mu > 0 \text{ and } v = E^T \mu
\end{align*}
\]

### IV. Refining Lyapunov Functions

Verifying the asymptotic stability of the origin relies on showing that the candidate Lyapunov function decreases along all trajectories passing through every point in a neighborhood of the origin. The feasibility problem (8)-(12) performs this check over subsets of the state space. The core of our refinement algorithm is based on the idea that if a candidate Lyapunov function does not decrease along all trajectories at every point in a cell, we may be able to precisely split the cell into two cells where the Lyapunov function decreases at all points in one of those cells. Successive refinements using this principle may lead to identification of a valid PWL Lyapunov function for the system if one exists. The rest of this section makes the statements above precise and presents the refinement algorithm (in Section IV-C).

If the trajectories of \( \Omega_P \) do not decrease along any valid set of parameters \( \{ p_i \}_{i \in I_P} \) corresponding to a continuous
positive definite PWL candidate Lyapunov function $V_P(x)$, then (10) and (11) will be infeasible for some cells in $Q$. To simplify the exposition, throughout this section we assume that $\Omega_P$ satisfies the following assumptions:

A2 $a_{ij} = 0$ for all $i \in I_P$ and $j \in I_A$.
A3 $|I_A| = 1$ for all cells $X_i \in \mathcal{P}$.

Therefore, $A_i(x) = A_i x$ for each $i \in I_P$, and constraints (10) and (11) are simplified to

$$E_i^T \nu_i = -A_{Q \rightarrow P(i)} \pi_i, \quad \forall i \in I_Q, \text{ and } \nu_i \geq 1, \quad \forall i \in I_Q. \tag{14}$$

A. Identifying Cells To Refine

Let $\Omega_P$ be a piece-wise affine differential inclusion satisfying assumptions A2 and A3 (and therefore A1) and let $Q$ be a refinement of $\mathcal{P}$ with abstraction function $\pi_{Q \rightarrow P}$. The index set of $Q$ is $I_Q$. We define index sets $I_{dec}(Q, \mathcal{P})$ and $I_{cont}(Q)$ as was done in Section III. The dynamics associated with cell $Z_i \in I_Q$ is $A_i x$ where $k = \pi_{Q \rightarrow P(i)}$.

Equation (3) shows that (13) and (14) together are equivalent to the requirement for each cell $Z_i \in Q$,

$$-p_i^T A_{Q \rightarrow P(i)} \in \text{Int}(Z^*_i). \tag{15}$$

Let $Z_i \in Q \in \text{Ref}(\mathcal{P})$ be a cell for which (15) does not hold. Then $p_i^T A_{Q \rightarrow P(i)} + \nu_i^2 E_i \neq 0$ for any vector $\nu > 0$.

For a system $\Omega_P$, we define variable $s_i^Q$ given by

$$s_i^Q(p, \nu) = E_i^T \nu + A_{Q \rightarrow P(i)} \pi_i, \tag{16}$$

$$t_{ij} = p_i - p_j - \lambda_{ij} \eta_{ij}, \tag{17}$$

$$d_i^Q(p, \nu) = \min_{\nu \geq \nu_i} ||s_i^Q(p, \nu)||, \tag{18}$$

where $\nu_i$ is a constant vector associated with each cell in partition $Q$. Clearly by Lemma 2,

$$-p_i^T A_{Q \rightarrow P(i)} \in \text{Int}(Z^*_i) \iff \exists \tilde{\nu}_i > 0 \text{ such that } d_i^Q(p_i, \tilde{\nu}_i) = 0. \tag{19}$$

Therefore, the quantity $d_i^Q(p_i, \tilde{\nu}_i)$ becomes an indicator for the infeasibility of constraints (13) and (14). To compute this indicator, we define an objective function as

$$J_{\Omega_P, Q} = \sum_{i \in I_Q} ||s_i^Q(p, \nu)||, \tag{20}$$

where $s_i^Q(p, \nu)$, given by (16), is like a slack variable for constraint (13) for each $i \in I_Q$.

For each $Z_i \in Q$ where $Z_i = \{ x \in \mathbb{R}^n : E_i x \geq 0 \}$, we define $\tilde{\nu}_i$ as a vector such that

$$\tilde{\nu}_i > 0, \text{ and } E_i^T \tilde{\nu}_i = F_{Q \rightarrow P(i)}^T 1, \tag{21}$$

which always exists when $X_{Q \rightarrow P(i)} \setminus \{0\}$ is non-empty. Defining $\tilde{\nu}_i$ as in (21) allows comparison of $d_i^Q(p_i, \tilde{\nu}_i)$ and $d_i^R(p_i, \tilde{\nu}_i)$ when $R \in \text{Ref}(\mathcal{P})$. We minimize objective function (20) subject to constraints (8)-(12) to obtain the full (convex) optimization problem below.

$$\begin{align*}
\min_{p_i, \mu_i, \nu_i, \lambda_{ij}} & \quad J_{\Omega_P, Q} \\
\text{s.t.} & \quad p_i = E_i^T \mu_i, \quad \forall i \in I_Q, \\
& \quad \mu_i \geq \tilde{\nu}_i, \quad \forall i \in I_Q, \\
& \quad \nu_i \geq \bar{\nu}_i, \quad \forall i \in I_Q, \\
& \quad p_i - p_j \leq \lambda_{ij} \eta_{ij}, \quad \forall(i, j) \in I_{cont}(Q). \tag{22}
\end{align*}$$

If the optimal solution is given by $p^*_i, \mu^*_i, \nu^*_i$, and $\lambda^*_i$ over the usual index sets, then the objective value $J_{\Omega_P, Q}^*$ at this optimum solution is

$$J_{\Omega_P, Q}^* = \sum_{i \in I_Q} d_i^Q(p^*_i, \tilde{\nu}_i). \tag{27}$$

According to (19), cells $Z_i \in Q$ such that $d_i^Q(p^*_i, \tilde{\nu}_i) = ||s_i^Q(p^*_i, \nu^*_i)|| \neq 0$ may need to be refined. The (convex) optimization problem (22)-(26) has the following properties by construction:

Proposition 3. The optimization problem (22)-(26) always has a feasible solution.

Proposition 4. If the optimal value of (22)-(26) is zero then the equilibrium of $\Omega_P$ is asymptotically stable.

Proposition 4, fact (19) and equation (27) motivate us to decrease $J_{\Omega_P, Q}^*$. One way to achieve this objective is by refining $Q$. The characterization of an appropriate refinement method is the focus of the next section.
B. Refinement Step

The objective of our refinement step is to split $Z_i \in \mathcal{Q}$ into cells $Z_j, Z_k$, thereby refining $\mathcal{Q}$ into $\mathcal{R}$ such that either $d^Q_j(p^*_i(\mathcal{Q}), \bar{v}_j)$ or $d^Q_k(p^*_i(\mathcal{Q}), \bar{v}_k)$ equals zero, where $p^*_i(\mathcal{Q})$ corresponds to the optimal solution of (22)-(26) for PWL Lyapunov function $V_\mathcal{Q}(x)$. Furthermore, we want both $Z_j$ and $Z_k$ to be polyhedral cones with apices at the origin. Therefore, the refinement step must involve defining a hyperplane through the origin that divides a polyhedral cone $Z$ into two polyhedral cones $Z_1$ and $Z_2$. Figure 3 depicts an example of such a refinement step. The existence of a hyperplane that achieves our refinement objective is characterized by the result below.

Lemma 5. Let $Z = \{x \in \mathbb{R}^n : Ex \geq 0 \}$ where $E \in \mathbb{R}^{m \times n}$ and $r(E) = n$. Suppose $v \notin \text{Int}(Z^*)$. There exists $Z_1 = \{x \in \mathbb{R}^n : E_1 x \geq 0 \}$ and $Z_2 = \{x \in \mathbb{R}^n : E_2 x \geq 0 \}$ such that

i) $\{Z_1, Z_2\} \in \text{Ref}(Z)$, and

ii) $v \notin \text{Int}(Z^*)$

if and only if $v \notin Z^*$.

Proof. Sufficiency: We demonstrate sufficiency by construction, which serves as a procedure for computing the refinement. The set $Z^*$ is $-Z^*$ and is given by $Z^* = \{y \in (\mathbb{R}^n)^* : \mu^T E, \mu \leq 0 \}$. If $v \notin Z^*$, then $v \in (\mathbb{R}^n)^* \setminus Z^*$, which is given by

$$(\mathbb{R}^n)^* \setminus Z^* = \{y \in (\mathbb{R}^n)^* : J(y, E) > 0\},$$

where

$$J(y, E) = \min_{\mu \in \mathbb{R}^m : y = \mu^T E} \max_{i} \mu_i.$$  

Let $v = \mu^T E$.

$$I_+ = \{i \in \{1, 2, \ldots, m\} : \mu_i > 0\},$$

$$I_- = \{i \in \{1, 2, \ldots, m\} : \mu_i < 0\},$$

$$I_0 = \{i \in \{1, 2, \ldots, m\} : \mu_i = 0\}.$$  

We define vectors $E^+ = \sum_{i \in I_+} \mu_i E_i$, $E^- = \sum_{i \in I_-} \mu_i E_i$ and $E^0 \in \text{pos}_{>0} \left(\sum_{i \in I_0} \mu_i E_i\right)$. Therefore, $v = E^+ + E^-$. We want to define a halfplane parametrized by $E^{\text{new}} \in (\mathbb{R}^n)^*$ such that $E^{\text{new}} \notin X^0$ and $v$ can be written as a positive combination of $E^+, -E^-, E^0$ and $E^{\text{new}}$. Therefore, we need

$$v = E^+ + E^- = E^{\text{new}} + \alpha E^+ - \beta E^- + \gamma E^0, \quad (33)$$

where $\alpha, \beta, \gamma > 0$. This requirement implies that

$$E^{\text{new}} = (1 - \alpha)E^+ + (1 + \beta)E^- - \gamma E^0. \quad (34)$$

Since $v \notin Z^*$, we have that $|I_+| \geq 1$, so that $E^+ \neq 0$ and $E^+ \notin Z^*$. Therefore, to ensure that $E^{\text{new}} \notin Z^*$, we need $\alpha < 1$.

To summarize, if $v \notin Z^*$ we can define $E^{\text{new}}$ such that $E^{\text{new}} \notin Z^*$ and

$$v \in \text{pos}_{>0} (\{E^{\text{new}}\} \cup \{E^i\}_{i \in \{1, 2, \ldots, m\}}).$$

A consequence of (35) is that if we define

$$Z_1 = \{x \in \mathbb{R}^n : E_1 x \geq 0\}, \quad \text{and} \quad (36)$$

$$Z_2 = \{x \in \mathbb{R}^n : E_2 x \geq 0\}, \quad \text{and} \quad (37)$$

then

i) $\{Z_1, Z_2\} \in \text{Ref}(Z)$, and

ii) $v \notin \text{Int}(Z^*)$.

Necessity:

Since $\{Z_1, Z_2\} \in \text{Ref}(Z)$, and $Z_1, Z_2$ are polyhedral cones, we can represent $Z_1$ by the inequalities $E_1 x \geq 0$ and $e^T x \geq 0$ and $Z_2$ by the inequalities $E_2 x \geq 0$ and $e^T x \leq 0$, where $r(E) = n$ and $e \notin Z^*$ (otherwise $Z_1$ or $Z_2$ will be empty).

Let $v \in \text{Int}(Z^*)$. Then $v$ always has representation as a positive combination of $e$ and all rows of $E$, say $v = \nu^T E + \eta e$, where $\nu > 0, \eta > 0$. Since $e \notin Z^*$, then any solution of $e = \mu^T E$ has $\max_i \mu_i > 0$. Therefore, we can represent $v$ as

$$v = \nu^T E + \eta e$$

$$= (\nu^T + \eta \mu^T) E = \xi^T E. \quad (39)$$

Since $\nu > 0, \eta > 0$ and $\max_i \mu_i > 0$, we must have that $\max_i \xi_i > 0$. Therefore, $\xi \notin Z^*$.

Thus, if $v \notin Z^*$ we can explicitly construct a partition $\{Z_1, Z_2\}$ of $Z$ such that $v \in \text{Int}(Z^*)$. The next result provides a condition for checking whether $v \in (\mathbb{R}^n)^*$ satisfies $v \in Z^*$ or not.

Lemma 6. Let $Z = \{x \in \mathbb{R}^n : Ex \geq 0\}$ where $E \in \mathbb{R}^{m \times n}$ is full rank, $v \in Z^*$ if and only if there exists $\mu \geq 0$ such that $v + \mu^T E = 0$.

If $-p^T A \in \text{Int}(Z^*)$ for a cell $Z_i \in \mathcal{Q}$ with corresponding dynamics $\dot{x} = Ax$, then there exists $\bar{v} > 0$ such that $d^Q_i(p, \bar{v}) = 0$. However, the constant $\bar{v}$ given in (21) may be such that $d^Q_i(p, \bar{v}) \neq 0$. Therefore, to refine a cell $Z_i \in \mathcal{Q}$ into $Z_j, Z_k$ such that either $d^Q_j(p^*_i, \bar{v}_j)$ or $d^Q_k(p^*_i, \bar{v}_k)$ equals zero as mentioned earlier, we must have that

$$(-p^*_i)^T A_{\mathcal{Q} \rightarrow \mathcal{P}(i)} - 1^T F_{\mathcal{Q} \rightarrow \mathcal{P}(i)} \notin Z^0. \quad (40)$$

We refer to the method of refinement in Lemma 5 as a $d$-split refinement, since it affects the quantity $d^Q_i$ in (18) in the way shown in the next below. The partition it induces is called a $d$-split partition.

Lemma 7. Let $\Omega_{\mathcal{P}}$ be a piecewise affine dynamical system and $V_{\mathcal{Q}}$ be a piecewise linear Lyapunov function where $\mathcal{Q} \in \text{Ref}(\mathcal{P})$. Let $p^* \in \mathbb{R}^n$ be the optimal value of the parameter $p_i$ for cell $Z_i \in \mathcal{Q}$ corresponding to the solution of (22)-(26) for Lyapunov function $V_{\mathcal{P}}(x)$ such that $p^*$ satisfies (40) but not (15). Let $Z_j$ and $Z_k$ be a $d$-split refinement of $Z_i \in \mathcal{Q}$ such that $p^* \notin \text{Int}(Z_j^*)$ and $p^* \notin \text{Int}(Z_k^*)$. Then $p^* \notin \text{Int}(Z_i^*)$ as well.
Lemma 7. Then, the following result.

Proof. The fact that $d^R_k(p^*, \bar{v}_j) = 0$ holds by a direct application of Lemma 5 and (19). The fact that $d^R_j(p^*, \bar{v}_j) = d^Q_j(p^*, \bar{v}_j)$ can be shown using the construction of $E^{\text{new}}$ in Lemma 5, which we omit due to space constraints.

The value of a $d$-split refinement is that it allows us to strictly reduce the set of points for which the current PWL Lyapunov function is non-decreasing, without increasing the objective function (20). This property is characterized by the following result.

Theorem 8. Let $\Omega_\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ satisfy the conditions of Lemma 7. Then,

\[ J_{\Omega_\mathcal{P}, \mathcal{Q}} \leq J_{\Omega_\mathcal{P}, \mathcal{Q}}. \]

Proof. Given a Lyapunov function $V_\mathcal{Q}(x)$, we can solve (22)-(26) to obtain an optimal solution $\{p^*_i\}$, $\{\mu^*_i\}$, $\{\nu^*_i\}$, and $\{\lambda^*_j\}$. We denote $p^*_i$ corresponding to a Lyapunov function $V_\mathcal{Q}(x)$ as $p^*_i(\mathcal{Q})$ when the partition is not clear. We have that

\[ J_{\Omega_\mathcal{P}, \mathcal{Q}} = \sum_{i \in \mathcal{I}_\mathcal{Q}} d^Q_i(p^*_i, \bar{v}_i). \]

Due to Lemma 7,

\[ d^R_j(p_j, \bar{v}_j) + d^R_k(p_k, \bar{v}_k) = d^Q(p^*_i, \bar{v}_i). \]

In turn,

\[ \sum_{j \in \mathcal{I}_\mathcal{R}} d^R_j(p^*_j, \bar{v}_j) = J_{\Omega_\mathcal{P}, \mathcal{Q}}. \]

Since $\mathcal{R}$ is a refinement of $\mathcal{Q}$, the optimal parameters $p^*_i(\mathcal{Q})$ of (22)-(26) with Lyapunov function $V_\mathcal{Q}(x)$ can be mapped to a set of parameters $\{p_j\}_{j \in \mathcal{R}}$ that correspond to a feasible solution of (22)-(26) with Lyapunov function $V_\mathcal{R}(x)$ using the abstraction function $\pi_{\mathcal{R} \rightarrow \mathcal{Q}}$. Therefore

\[ J_{\Omega_\mathcal{P}, \mathcal{R}} \leq \sum_{j \in \mathcal{I}_\mathcal{R}} d^R_j(p^*_j, \bar{v}_j). \]

Combining (44) and (45) completes the proof.

Theorem 8 motivates us to use the $d$-split refinement in Lemma 5 to develop our refinement algorithm, given in the next section.

We now show when a $d$-split leads to a precise reduction.

Theorem 9. Let $\Omega_\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ satisfy the conditions of Lemma 7. Then,

\[ J_{\Omega_\mathcal{P}, \mathcal{R}} < J_{\Omega_\mathcal{P}, \mathcal{Q}}. \]

C. Refinement Algorithm

Our refinement algorithm involves a sequence of optimization problems of the form (22)-(26) corresponding to a sequence of partitions $\mathcal{R}_k$ where $\mathcal{R}_{k+1} \in \text{Ref}(\mathcal{R}_k)$ and $\mathcal{R}_0 = \mathcal{P}$. The refinement of $\mathcal{R}_k$ involves searching for a cell $Z_i \in \mathcal{R}_k$ for which constraint (13) is infeasible ($d(p^*_i(\mathcal{R}_k), \bar{v}_i) \neq 0$) but can be split into two cells ($p^*_i(\mathcal{R}_k)$ satisfies (40)) so that one of them will satisfy a constraint of the form (13) corresponding to that cell. This refinement step strictly reduces the set of points for which the Lyapunov function at iteration $k$ is non-decreasing. Algorithm 1 describes this procedure. We can show the following properties.

Proposition 10. Algorithm 1 is sound.

Proof. The algorithm terminates when the optimization problem (22)-(26) has optimal value zero. By Proposition 4, which relies on Lemma 1, the Lyapunov function returned by Algorithm 1 verifies asymptotic stability of the origin. □

Remark 2 (On Completeness). For Algorithm 1 to be complete, if a PWL Lyapunov function exists for the system then the system is asymptotically stable and Algorithm 1 must find a valid PWL Lyapunov function. Therefore, there must exist $K \in \mathbb{N}, K < \infty$ such that $J_{\Omega_\mathcal{P}, \mathcal{R}_k} = 0$.

Since Algorithm 1 uses $d$-split refinements, we know that the objective function is non-increasing. If the algorithm never terminates, there exists $K \in \mathbb{N}, K < \infty$ such that $J_{\Omega_\mathcal{P}, \mathcal{R}_k} = \bar{J} > 0$ for all $k \geq K$. This situation implies that the candidate Lyapunov function remains the same at all points in the state space over successive iterations. The successive partitions produced by the refinement algorithm are either obtained by only refining cells produced by the preceding refinement step, or are the same partition as that in the preceding step (no cells are refined). We conjecture that neither situation, in which the Lyapunov function at each
point remains the same for all successive iterations, can occur for asymptotically stable systems when using Algorithm 1. However, such a property remains to be shown.

It may be possible that refining cells in \( I_R \setminus I_{\text{refine}} \) ensures that the objective strictly decreases, since for the refined cells the quantity \( d_i^{R_{i+1}} \) will remain zero, however the increased number of pieces allows more flexibility in the parameters of the Lyapunov function for each cell. Again, a precise characterization is not available at the time of writing this paper.

V. Examples

We present a number of piece-wise affine dynamical systems and demonstrate the benefits of the refinement-based search for a piece-wise linear Lyapunov function proposed in Section IV.

Example 1 (Based on Branciky’s example [10]). This classic example involves a state-based switched system with two modes, where the dynamics are neutrally stable in each mode. Appropriate switching makes the origin asymptotically stable.

\[
\Sigma_1 : \dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x \\
\Sigma_2 : \dot{x} = \begin{bmatrix} 0 & 1 \\ -0.1 & 0 \end{bmatrix} x
\]

(47)

\[X_1 = \{ x \in \mathbb{R}^2 | x_1 x_2 \geq 0 \} \]

\[X_2 = \{ x \in \mathbb{R}^2 | x_1 x_2 < 0 \} \]

Algorithm 1 finds a valid Lyapunov function with 9 cells in 2.386 seconds.

Example 2 (Prajna et al. [11]). Arbitrary Switching. The system arbitrarily switches between two modes

\[\dot{x} = A_1 x = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix} x, \quad \text{and} \quad \dot{x} = A_2 x = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix} x.\]

The differential inclusion

\[\dot{x} \in \text{conv} \ (A_1 x, A_2 x)\]

(48)

captures this arbitrarily switching between the two modes.

Prajna et al. [11] use sums-of-squares (SOS) optimization to find a polynomial Lyapunov function of degree 6 for this system. The time taken is not reported. SOS-based methods typically scale poorly with degree. The dashed blue line in Figure 5 depicts a level set of this polynomial.

Algorithm 1 finds a valid Lyapunov function with 66 cells in 5.499 seconds. Figure 5 depicts a level set of this polynomial by the solid red line. Note that it approximates the contour (dashed blue line) obtained in [11].

Example 3 (State-based 3D Switched System). We convert the two dimensional system in Example 1 into a three dimensional system by adding a third variable with stable linear dynamics decoupled from the first two. The dynamical
We presented some examples to demonstrate the utility of this algorithm. While the algorithm succeeds in finding PWL Lyapunov functions in these examples, the time taken may scale poorly with dimension. While we are sure that each refinement step does not increase the value of the objective function involved in our iterative algorithm, our algorithm does not guarantee decrease in the objective at any iteration. This lack prevents deciding if the algorithm is complete or not. Furthermore, if the size of the partition increases too quickly relative to any decrease in the objective function, the computational time can increase significantly. Future work will involve developing methods to improve the refinement procedure by identifying refinement steps that may not decrease the objective function and avoiding refinement of those cells. This approach may also enable showing completeness of the existing algorithm or modifying it to obtain a complete algorithm.

VI. DISCUSSION AND FUTURE WORK

We have presented an automatic algorithm for finding PWL Lyapunov functions for verifying the asymptotic stability of piece-wise affine differential inclusions. The algorithm uses a sequential optimization approach, where iteration involves refining cells in a precise manner. If the PWL Lyapunov function can be refined, then the set of points in the state space at which the Lyapunov function is non-decreasing along trajectories strictly decreases.