ME-599/699
Exam 1 – March 11, 2004

Problem 1 – Score____ (25 Points Possible)
Problem 2 – Score____ (25 Points Possible)
Problem 3 – Score____ (25 Points Possible)
Problem 4 – Score____ (15 Points Possible)
Problem 5 – Score____ (10 Points Possible)

Total - Score____ (100 Points Possible)

• This exam is closed book and closed notes.
• Calculators are allowed.
• There is some information on the final pages that may be of help.
1. (25 pts.) The input to the system shown is the external applied force, $F$. Use the finite difference method to determine the unit step response, $x(t)$. Use a time step, $\Delta t$, of 0.5 seconds, and find $x(t)$ for $t=0.5$ seconds, 1.0 seconds, and 1.5 seconds. Fill in the table below with your solution, and also show your calculations.

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>$x(t)$ (meters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1250</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2917</td>
</tr>
<tr>
<td>1.5</td>
<td>0.4167</td>
</tr>
</tbody>
</table>

For unit step response, the system equation is:

$$\ddot{x} + 2\dot{x} + 2x = 1; \quad x(0) = \dot{x}(0) = 0$$

$$\chi_i = \dot{x}_i = 0; \quad h = 0.5; \quad m = 1; \quad B = 2; \quad K = 2; \quad F = 1 \text{ for } t > 0.$$

$$\chi_2 = \chi_i + h\dot{\chi}_i + \frac{h^2}{2} \left[ \frac{1}{m} \left[ -K\chi_i - B\dot{\chi}_i + F(0) \right] \right] = 0 + 0.5(0) + \frac{0.5^2}{2} \left[ -2(0) - 2(0) + 1 \right]$$

$$\chi_2 = 0.125 \, \text{m}$$

$$\dot{\chi}_2 = \frac{1}{2m+8h} \left\{ \frac{2m}{h} \left[ \chi_2 - \chi_i \right] + h \left[ -K\chi_2 + F(0) \right] \right\} = \frac{1}{2(1)+2(0.5)} \left\{ \frac{2(1)}{0.5} \left[ 0.125 - 0 \right] + 0.5 \left[ -2(0.125) + 1 \right] \right\}$$

$$\dot{\chi}_2 = 0.2917 \, \frac{\text{m}}{\text{s}}$$

$$\chi_{i+1} = 2\chi_i - \chi_{i-1} + \frac{h^2}{m} \left[ -K\chi_i - B\dot{\chi}_i + F_a(t) \right]$$

$$\dot{\chi}_{i+1} = \frac{1}{2m+8h} \left\{ \frac{2m}{h} \left[ \chi_{i+1} - \chi_i \right] + h \left[ -K\chi_{i+1} + F_a(t_{i+1}) \right] \right\}$$

$$\chi_{i+1} = 2\chi_i - \chi_{i-1} - 0.5\chi_i - 0.5\dot{\chi}_i + 0.25$$

$$\dot{\chi}_{i+1} = \frac{1}{3} \left\{ 4 \left[ \chi_{i+1} - \chi_i \right] - \chi_{i+1} + 0.5 \right\}$$

$$\chi_3 = 2(0.125) - 0 - 0.5(0.125) - 0.5(0.2917) + 0.25 = 0.2917$$

$$\dot{\chi}_3 = \frac{1}{3} \left\{ 4 \left[ 0.2917 - 0.125 \right] - 0.2917 + 0.5 \right\} = 0.2917$$

$$\chi_4 = 2(0.2917) - 0.125 - 0.5(0.2917) - 0.5(0.2917) + 0.25 = 0.4167$$

$$\dot{\chi}_4 = 0.2917 \, \frac{\text{m}}{\text{s}}$$
2. (25 pts.) Consider the massless, nonlinear system below. Assume $K=5$ and $F=2$, in some consistent system of units. Perform 2 iterations of the Newton-Raphson method to find the static displacement. In the table, list the displacement, $x$, and the force imbalance at the end of each iteration. Show all of your work. Assume the initial displacement is $x_0=0$.

\[
\begin{array}{|c|c|c|}
\hline
\text{Iteration} & \text{x} & \text{Force Imbalance} \\
\hline
1 & 0.4000 & 0.160 \\
2 & 0.4381 & 0.0014 \\
\hline
\end{array}
\]

\[
Kx \quad \rightarrow F 
\]

\[
\Sigma F=0 \Rightarrow F=Kx
\]

\[
F = (5-x) x = 5x - x^2
\]

\[
K_t = \frac{dF}{dx} = 5 - 2x
\]

\[
\chi_1 = K_t^{-1} (F_{\text{app}} - F_{\text{spring}}) + x_0
\]

\[
x_0 = 0 \Rightarrow K_t \bigg|_{x_0} = 5 - 2(0) = 5 \quad \Rightarrow (F_{sp})_0 = Kx_0 = 0
\]

\[
\chi_1 = \frac{1}{5} (2 - 0) + 0 = 0.4
\]

\[
(F_{sp})_{x=x_1} = Kx_1 = (5 - 0.4) 0.4 = 1.84
\]

\[
(\text{Force Imbalance})_{x=x_1} = 2 - 1.84 = 0.16
\]

\[
(K_t)_{x=x_1} = 5 - 2(0.4) = 4.2
\]

\[
\chi_2 = \frac{1}{4.2} (0.16) + 0.4 = 0.4381
\]

\[
(F_{sp})_{x=x_2} = Kx_2 = (5 - 0.4381) (0.4381) = 1.9986
\]

\[
(\text{Force Imbalance})_{x=x_2} = 2 - 1.9986 = 0.0014
\]
3. (25 pts.) The system below consists of an aluminum bar, a steel bar, a spring, and a lumped mass. Assume horizontal motion only, and use the finite element method, based on the node numbering shown, to write a single second order differential equation for horizontal motion of the system. Your final equation should be expressed in SI units. Use a single bar finite element for the aluminum bar and a single bar finite element for the steel bar, with a consistent mass formulation. You will also need a spring element and a lumped mass element. Note that this is a one-dof system when only horizontal motion is considered (see $d_{2x}$ shown in the figure). Nodes 1 and 3 are constrained from motion. State the undamped natural frequency in units of Hz, for horizontal motion of node 2. Properties are provided in the table below.

<table>
<thead>
<tr>
<th></th>
<th>Aluminum Bar</th>
<th>Steel Bar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>168 lb/ft$^3$</td>
<td>487 lb/ft$^3$</td>
</tr>
<tr>
<td>Modulus of Elasticity</td>
<td>$1.44E9$ lb/ft$^2$</td>
<td>$4.32E9$ lb/ft$^2$</td>
</tr>
<tr>
<td>Cross-Sectional Area</td>
<td>$650$ mm$^2$</td>
<td>$750$ mm$^2$</td>
</tr>
</tbody>
</table>

After applying B.C.'s:

$$d_{ix} = d_{3x} = d_{2x}, \quad e_{g\'s} \text{ reduce to:}$$

$$M_{22} \ddot{d}_{2x} + K_{22} \dot{d}_{2x} = F_2$$

$$M_{22} = \frac{P_{ae} \frac{A_{ae} L_{ae}}{L_{ae}} + E_{st} \frac{A_{st} L_{st}}{3}}{3} + M_2$$

$$K_{22} = \frac{A_{ae} E_{ae}}{L_{ae}} + K_{spring} + \frac{A_{st} E_{st}}{L_{st}}$$

$$P_{ae} = (168 \frac{lb}{ft^3}) \left( \frac{1 \text{ Kg}}{2.2 \text{ lb}} \right) \left( \frac{1 \text{ ft}^3}{12 \text{ in}^3} \right) \left( \frac{3.927 \text{ m}^3}{1 \text{ m}^3} \right) = 2.76 \frac{\text{Kg}}{\text{m}^3}$$

$$P_{st} = \left( \frac{487}{168} \right) \frac{P_{ae}}{\text{m}^3} = 7758 \frac{\text{Kg}}{\text{m}^3}$$

$$A_{ae} = 6.50 \text{ mm}^2 \left( \frac{0.001 \text{ m}^2}{1 \text{ mm}^2} \right) = 0.000065 \text{ m}^2$$

$$A_{st} = 0.00075 \text{ m}^2$$

$$E_{ae} = (1.44E9 \frac{\text{lb}}{\text{ft}^2}) \left( \frac{1 \text{ N}}{2.2 \text{ lb}} \right) \left( \frac{1 \text{ ft}^2}{12 \text{ in}^2} \right) \left( \frac{3.927 \text{ m}^2}{1 \text{ m}^2} \right) = 6.86 \times 10^8 \frac{\text{N}}{\text{m}^2}$$

$$E_{st} = \frac{432}{144} \left( \frac{E_{ae}}{\text{m}^2} \right) = 20.6 \times 10^8 \frac{\text{N}}{\text{m}^2}$$

Lumped mass at node 2: 1 lb $\left( \frac{1 \text{ Kg}}{2.2 \text{ lb}} \right) = 0.455 \text{ Kg}$

$$L_{ae} = 5 \text{ in} \left( \frac{1 \text{ m}}{39.27 \text{ in}} \right) = 0.127 \text{ m}$$

$$L_{st} = 0.2 \text{ m}$$
\[ M_{22} = \left[ \frac{(2670)(0.00065)(0.127)}{3} + \frac{(7758)(0.00075)(0.2)}{3} + 0.455 \right] \text{ kg} \]

\[ M_{22} = 0.916 \text{ kg} \]

\[ K_{22} = \left[ \frac{(0.00065)(6.86E10)}{0.127} + 1E8 + \frac{(0.00075)(20.6E10)}{0.2} \right] \frac{N}{m} \]

\[ K_{22} = 1.22E9 \frac{N}{m} \]

\[ F = 10 \text{ lb} \times \left( \frac{1N}{0.2248 \text{ lb}} \right) = 44.48 \text{ N} \]

\[ 0.916 \ddot{z}_2 + 1.22E9 \ z_2 = 44.48 \]

\[ \omega_n = \sqrt{\frac{1.22E9}{0.914}} \ \frac{\text{rad}}{\text{sec}} = 36495 \ \frac{\text{rad}}{s} \]

\[ f_n = 5808 \ \text{Hz} \Rightarrow \text{Undamped Natural Frequency} \]
4. (15 pts.) Consider the three thin cantilevered beams shown, in a gravitational field, with gravity acting downward (acceleration due to gravity is in the \(-y\) direction). All three beams have identical geometries (identical lengths, cross-sections, and moments of inertia for bending about the \(z\)-direction) and identical material properties. They are all loaded by a constant point force, \(F\), directed to the right (+x direction) at their unconstrained ends. Beams 2 and 3 each have a lumped mass, \(M\), at their unconstrained ends. Beam 3 is inverted with respect to the other two. Assume the weight of the mass is not sufficient to produce buckling of Beam 3, but it is sufficient to significantly affect the axial stresses in Beam 2 and Beam 3.

a) (5 pts.) If a static finite element analysis is performed on all three beams, based on linear, small deflection assumptions (no stress stiffening effects included), do you expect a difference in the calculated \(x\)-direction deflection at the loaded ends for the three beams? If so, which beam will have the largest \(x\)-direction deflection, and which will have the smallest at the loaded end? Explain your answer.

No difference because there is no stress stiffening in the analysis, so all 3 beams have identical lateral stiffness.

b) (5 pts.) Based on the analysis from Case (a), do you expect a difference in the calculated \(y\)-direction deflection at the unconstrained ends between Beam 1 and Beam 2? If so, which of the two will have the larger \(y\)-direction deflection? Explain your answer.

With gravity loading included, the weight of the lumped mass will cause a greater deflection in beam 2, in the vertical \((y)\) direction.

c) (5 pts.) If a static finite element analysis is again performed on all three beams, but it is based on nonlinear, large deflection assumptions (stress stiffening effects included), do you expect a difference in the calculated \(x\)-direction deflection at the loaded ends for the three beams? If so, which beam will have the largest \(x\)-direction deflection and which will have the smallest at the loaded end? Explain your answer.

Greatest deflection: Beam 3
Smallest deflection: Beam 2

Beam 2 has tensile stress, therefore stress stiffening increases its lateral stiffness. Beam 1 will have axial tension due to its own weight, but less than Beam 1, so less lateral stiffness. Beam 3 has even less lateral stiffness due to axial compressive stress (softening).
5. (10 pts.) Consider a bar, or “plate”, with a hole which has uniform thickness “into” the page. It is in tension, as shown, with uniform pressure, \( P \), applied at each end. Assume it is made of the material with stress-strain curve shown below. Also, assume the width, \( W \), is five times the diameter of the hole.

<table>
<thead>
<tr>
<th>Pressure, ( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W )</td>
</tr>
<tr>
<td>Pressure, ( P )</td>
</tr>
</tbody>
</table>

- Stress
  - 650 MPa

\[ E = \frac{650 \times 10^6}{0.008} = 8.125 \times 10^8 \, \text{N/m}^2 \]

a) (5 pts.) What is the modulus of elasticity for the material?

b) (5 pts.) Assume two static finite element analyses are performed on the bar. For one analysis, the material is assumed to behave linearly, with constant modulus of elasticity. In the other analysis, the nonlinear material model was included. The material models were the only differences in the analyses. If the applied pressure, \( P \), is 100 MPa, would you expect any significant differences in stress, strain, and deflection results from the two finite element analyses? Explain your answer.

\[ \sigma = \frac{P W t}{(W-d)t^2} = \frac{P W}{(W-w)^2} \]

If \( t \) is thickness, \( d \) is hole diameter:

\[ \sigma = \frac{5PW}{4W} = \frac{5}{4} P = 1.25E6 \, \text{N/m}^2 \]

From Figure, \( K_t \approx 2.5 \) (see attachment)

Peak Stress \( \approx (2.5) (1.25E6) = 3.125 \, \text{MPa} \)

Yield Point \( = 650 \, \text{MPa} \)

Expect no significant difference because peak stress is below yield.
\[ x_2 = x_1 + h v_1 + \frac{h^2}{2} \left( \frac{1}{M} \left[ -K x_1 - B v_1 + f_a(t_1) \right] \right) \]

\[ v_2 = \left[ \frac{1}{2M + Bh} \right] \left\{ \frac{2M}{h} \left[ x_2 - x_1 \right] + h \left[ -K x_2 + f_a(t_2) \right] \right\} \]

\[ x_{i+1} = 2x_i - x_{i-1} + \frac{h^2}{M} \left[ -K x_i - B v_i + f_a(t_i) \right] \]

\[ v_{i+1} = \left[ \frac{1}{2M + Bh} \right] \left\{ \frac{2M}{h} \left[ x_{i+1} - x_i \right] + h \left[ -K x_{i+1} + f_a(t_{i+1}) \right] \right\} \]

\[ x_{i+1} = x_i + h v_{i-ave} \]

\[ v_{i+1} = v_i + h a_{i-ave} \]

\[ v_{i-ave} = \frac{1}{6} \left[ v_a + 4v_b + v_c \right] \]

\[ a_{i-ave} = \frac{1}{6} \left[ a_a + 4a_b + a_c \right] \]

---

Element stiffness matrix for a bar element with end nodes 1 and 2, oriented along x-axis:

\[
[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

Element consistent mass matrix for a bar element with end nodes 1 and 2, oriented along x-axis:

\[
[m] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\]

1 N = 0.2248 lb; 1 m = 39.27 inches; Acceleration due to gravity = 32.2 ft/s²
TABLE A-15
Charts of Theoretical Stress-Concentration Factors $K_t$

FIGURE A-15-1
Bar in tension or simple compression with a transverse hole. $\sigma_t = F/A$, where $A = (w - d)t$ and $t$ is the thickness.