Vector $\boldsymbol{\nabla}^2\mathbf{A}$ in Rectangular, Cylindrical, and Spherical Coordinates

In all coordinate systems, $\boldsymbol{\nabla}^2\mathbf{A}$ is given by the identity

$$\boldsymbol{\nabla}^2\mathbf{A} = \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} \ . \tag{1}$$

However, it is not necessarily the case that $\boldsymbol{\nabla}^2\mathbf{A} = \nabla^2\mathbf{A}$. The vector $\boldsymbol{\nabla}^2$ operator is derived for the three main coordinate systems. A summary of the results is given. For rectangular coordinates,

$$\boldsymbol{\nabla}^2\mathbf{A} = \nabla^2\mathbf{A} \ . \tag{2}$$

For cylindrical coordinates,

$$\begin{aligned}
\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_\rho - \frac{1}{\rho^2}\left[A_\rho + 2\frac{\partial A_\phi}{\partial\phi}\right] \\
\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_\phi - \frac{1}{\rho^2}\left[A_\phi - 2\frac{\partial A_\rho}{\partial\phi}\right] \\
\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_z \ .
\end{aligned}$$

Finally, for spherical coordinates,

$$\begin{aligned}
\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_r - \frac{2}{r^2}\left[A_r + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_\theta + \frac{1}{r^2}\left[-\frac{1}{\sin^2\theta}A_\theta + 2\frac{\partial A_r}{\partial\theta} - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_\phi + \frac{1}{r^2\sin\theta}\left[-\frac{1}{\sin\theta}A_\phi + 2\frac{\partial A_r}{\partial\phi} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right] \ .
\end{aligned} \tag{3}$$

1 Rectangular Coordinates

$$\boldsymbol{\nabla}^2\mathbf{A} = \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} \tag{4}$$

$$\boldsymbol{\nabla}^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \tag{5}$$

$$\boldsymbol{\nabla} w = \hat{\mathbf{x}}\frac{\partial w}{\partial x} + \hat{\mathbf{y}}\frac{\partial w}{\partial y} + \hat{\mathbf{z}}\frac{\partial w}{\partial z} \tag{6}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{7}$$

$$\boldsymbol{\nabla}\times\mathbf{A} = \hat{\mathbf{x}}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \hat{\mathbf{y}}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \hat{\mathbf{z}}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \tag{8}$$

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] \\
&+ \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] \\
&+ \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right] \qquad (9)\\
&= \hat{\mathbf{x}}\left[\frac{\partial^2 A_y}{\partial y\partial x} - \frac{\partial^2 A_x}{\partial y\partial y} - \frac{\partial^2 A_x}{\partial z\partial z} + \frac{\partial^2 A_z}{\partial z\partial x}\right] \\
&+ \hat{\mathbf{y}}\left[\frac{\partial^2 A_z}{\partial z\partial y} - \frac{\partial^2 A_y}{\partial z\partial z} - \frac{\partial^2 A_y}{\partial x\partial x} + \frac{\partial^2 A_x}{\partial x\partial y}\right] \\
&+ \hat{\mathbf{z}}\left[\frac{\partial^2 A_x}{\partial x\partial z} - \frac{\partial^2 A_z}{\partial x\partial x} - \frac{\partial^2 A_z}{\partial y\partial y} + \frac{\partial^2 A_y}{\partial y\partial z}\right] \qquad (10)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{A} &= \boldsymbol{\nabla}\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] \\
&= \hat{\mathbf{x}}\left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z}\right] \\
&+ \hat{\mathbf{y}}\left[\frac{\partial^2 A_x}{\partial y\partial x} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y\partial z}\right] \\
&+ \hat{\mathbf{z}}\left[\frac{\partial^2 A_x}{\partial z\partial x} + \frac{\partial^2 A_y}{\partial z\partial y} + \frac{\partial^2 A_z}{\partial z^2}\right] \qquad (11)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{x}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_x + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z} - \frac{\partial^2 A_y}{\partial y\partial x} - \frac{\partial^2 A_z}{\partial z\partial x} = \nabla^2 A_x \\
\hat{\mathbf{y}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_y + \frac{\partial^2 A_x}{\partial y\partial x} + \frac{\partial^2 A_z}{\partial y\partial z} - \frac{\partial^2 A_z}{\partial z\partial y} - \frac{\partial^2 A_x}{\partial x\partial y} = \nabla^2 A_y \\
\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \nabla^2 A_z + \frac{\partial^2 A_x}{\partial z\partial x} + \frac{\partial^2 A_y}{\partial z\partial y} - \frac{\partial^2 A_x}{\partial x\partial z} - \frac{\partial^2 A_y}{\partial y\partial z} = \nabla^2 A_z
\end{aligned}$$

2 Cylindrical Coordinates

$$\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \tag{12}$$

$$\nabla^2 w = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{\partial^2 w}{\partial z^2} \tag{13}$$

$$\nabla w = \hat{\boldsymbol{\rho}}\frac{\partial w}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial w}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial w}{\partial z} \tag{14}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{15}$$

$$\nabla \times \mathbf{A} = \hat{\boldsymbol{\rho}}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \hat{\mathbf{z}}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right] \tag{16}$$

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{A} = \hat{\boldsymbol{\rho}}&\left[\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\right] \\
+ \hat{\boldsymbol{\phi}}&\left[\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) - \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)\right] \\
+ \hat{\mathbf{z}}&\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\right) - \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\right]
\end{aligned} \tag{17}$$

$$\begin{aligned}
\nabla [\nabla \cdot \mathbf{A}] &= \nabla \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\right] \\
&= \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\right) \\
&+ \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\right) \\
&+ \hat{\mathbf{z}}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\right)
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\rho}} \cdot \nabla^2 \mathbf{A} &= \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho)\right) + \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi}\right) + \frac{\partial^2 A_z}{\partial \rho \partial z} \\
&- \left[\frac{1}{\rho^2}\frac{\partial}{\partial \phi}\left(\frac{\partial}{\partial \rho}(\rho A_\phi)\right) - \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial \phi^2} - \frac{\partial^2 A_\rho}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial \rho}\right] \\
&= \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho)\right) + \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial \phi^2} + \frac{\partial^2 A_\rho}{\partial z^2} \\
&+ \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi}\right) + \frac{\partial^2 A_z}{\partial \rho \partial z} - \frac{1}{\rho^2}\frac{\partial}{\partial \phi}\left(\frac{\partial}{\partial \rho}(\rho A_\phi)\right) - \frac{\partial^2 A_z}{\partial z \partial \rho} \\
&= \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho)\right) + \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial \phi^2} + \frac{\partial^2 A_\rho}{\partial z^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi}
\end{aligned} \tag{18}$$

$$
\begin{aligned}
\hat{\boldsymbol{\phi}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} &= \frac{1}{\rho^2} \frac{\partial^2}{\partial\phi\partial\rho} \left(\rho A_\rho\right) + \frac{1}{\rho^2} \frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{1}{\rho} \frac{\partial^2 A_z}{\partial\phi\partial z} \\
&- \left[\frac{1}{\rho} \frac{\partial^2 A_z}{\partial z\partial\phi} - \frac{\partial^2 A_\phi}{\partial z^2} - \frac{\partial}{\partial\rho} \left(\frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho A_\phi\right) \right) + \frac{\partial}{\partial\rho} \left(\frac{1}{\rho} \frac{\partial A_\rho}{\partial\phi} \right) \right] \\
&= + \frac{\partial^2 A_\phi}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{\partial}{\partial\rho} \left(\frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho A_\phi\right) \right) \\
&- \frac{1}{\rho} \frac{\partial^2 A_z}{\partial z\partial\phi} - \frac{\partial}{\partial\rho} \left(\frac{1}{\rho} \frac{\partial A_\rho}{\partial\phi} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial\phi\partial\rho} \left(\rho A_\rho\right) + \frac{1}{\rho} \frac{\partial^2 A_z}{\partial\phi\partial z} \\
&= \frac{\partial}{\partial\rho} \left(\frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho A_\phi\right) \right) + \frac{1}{\rho^2} \frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{\partial^2 A_\phi}{\partial z^2} + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial\phi}
\end{aligned} \tag{19}
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\begin{aligned}
\hat{\mathbf{z}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} &= \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho A_\rho\right) \right) + \frac{1}{\rho} \frac{\partial^2 A_\phi}{\partial z\partial\phi} + \frac{\partial^2 A_z}{\partial z^2} \\
&- \left[\frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial A_\rho}{\partial z} \right) - \frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial A_z}{\partial\rho} \right) - \frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial\phi^2} + \frac{1}{\rho} \frac{\partial^2 A_\phi}{\partial\phi\partial z} \right] \\
&= + \frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial A_z}{\partial\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial\phi^2} + \frac{\partial^2 A_z}{\partial z^2} \\
&+ \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho A_\rho\right) \right) + \frac{1}{\rho} \frac{\partial^2 A_\phi}{\partial z\partial\phi} - \frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial A_\rho}{\partial z} \right) - \frac{1}{\rho} \frac{\partial^2 A_\phi}{\partial\phi\partial z} \\
&= \nabla^2 A_z
\end{aligned} \tag{20}
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3 Spherical Coordinates

$$\nabla^2 \mathbf{A} = \nabla\left(\nabla\cdot\mathbf{A}\right) - \nabla\times\nabla\times\mathbf{A} \tag{21}$$

$$\nabla w = \hat{\mathbf{r}}\frac{\partial w}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial w}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} \tag{22}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{23}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right] \tag{24}$$

$$\begin{aligned}
\nabla\left(\nabla\cdot\mathbf{A}\right) &= \left[\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&= \hat{\mathbf{r}}\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&+ \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&+ \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]
\end{aligned} \tag{25}$$

$$\begin{aligned}
\nabla\times\nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}&\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\sin\theta\right) - \frac{\partial}{\partial\phi}\left(\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\right)\right] \\
+ \hat{\boldsymbol{\theta}}\frac{1}{r}&\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]\right) - \frac{\partial}{\partial r}\left(r\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\right)\right] \\
+ \hat{\boldsymbol{\phi}}\frac{1}{r}&\left[\frac{\partial}{\partial r}\left(r\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]\right)\right]
\end{aligned} \tag{26}$$

$$\begin{aligned}
\hat{\mathbf{r}}\cdot\nabla^2\mathbf{A} &= \frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&- \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\sin\theta\right) - \frac{\partial}{\partial\phi}\left(\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\right)\right]
\end{aligned} \tag{27}$$

$$\begin{aligned}
\hat{\boldsymbol{\theta}}\cdot\nabla^2\mathbf{A} &= \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&- \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]\right) - \frac{\partial}{\partial r}\left(r\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\right)\right]
\end{aligned} \tag{28}$$

$$\begin{aligned}
\hat{\boldsymbol{\phi}}\cdot\nabla^2\mathbf{A} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&- \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]\right)\right]
\end{aligned} \tag{29}$$

$$
\begin{aligned}
\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}^2\mathbf{A} &= \frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&-\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right)-\frac{\partial A_r}{\partial\theta}\right]\sin\theta\right)-\frac{\partial}{\partial\phi}\left(\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}\left(rA_\phi\right)\right]\right)\right] \\
&= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)\right)+\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)\right)+\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\
&-\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right)-\frac{\partial A_r}{\partial\theta}\right]\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}\left(rA_\phi\right)\right]\right) \\
&= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)\right)+\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)\right)+\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\
&-\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin\theta\frac{\partial}{\partial r}\left(rA_\theta\right)\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin\theta\frac{\partial A_r}{\partial\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}\right)-\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right.\right. \\
&= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}\right) \\
&-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial r}\left(rA_\theta\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_r}{\partial\phi}\right)-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial r}\left(rA_\phi\right)\right) \\
&= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_r}{\partial\phi}\right) \\
&-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial r}\left(rA_\theta\right)\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}\right)-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial r}\left(rA_\phi\right)\right) \\
&= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_r}{\partial\phi}\right) \\
&+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(rA_\theta\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}A_\theta\right)\right]\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\left[\frac{\partial}{\partial r}\left(\frac{1}{r}A_\phi\right)-\frac{1}{r^2}\frac{\partial}{\partial r}\left(rA_\phi\right)\right]\right) \\
&= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_r}{\partial\phi}\right) \\
&-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} &= \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&- \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]\right) - \frac{\partial}{\partial r}\left(r\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\right)\right] \\
&= \left[\frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\left(r^2 A_r\right)\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)\right] \\
&- \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_\theta}{\partial\phi}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right)\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial A_r}{\partial\theta}\right) \\
&= +\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rA_\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\phi^2} \\
&- \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right)\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial A_r}{\partial\theta}\right) + \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\left(r^2A_r\right)\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\
&= \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rA_\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\phi^2} \\
&+ \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\hat{\boldsymbol{\phi}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \\
&- \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]\right)\right] \\
&= \frac{1}{r^3\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi}{\partial\phi^2} \\
&- \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial r}\frac{\partial A_r}{\partial\phi} + \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rA_\phi\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial A_\theta}{\partial\phi} \\
&= +\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rA_\phi\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi}{\partial\phi^2} \\
&+ \frac{1}{r^3\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}\left(r^2A_r\right) - \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial r}\frac{\partial A_r}{\partial\phi} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) - \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial A_\theta}{\partial\phi} \\
&= +\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi}{\partial\phi^2} \\
&+ \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}
\end{aligned}
\tag{32}
$$

The final result is

$$\hat{\boldsymbol{\phi}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} = +\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(A_\phi \sin\theta\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin\theta}\frac{\partial A_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial \phi} \tag{33}$$

$$\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r A_\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_\phi}{\partial \phi} \tag{34}$$

$$\hat{\mathbf{r}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} = \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial A_r}{\partial \phi}\right) - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{35}$$

or

$$\hat{\boldsymbol{\phi}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} = \nabla^2 A_\phi + \frac{1}{r^2\sin\theta}\left[-\frac{1}{\sin\theta}A_\phi + 2\frac{\partial A_r}{\partial \phi} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial A_\theta}{\partial \phi}\right] \tag{36}$$

$$\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} = \nabla^2 A_\theta + \frac{1}{r^2}\left[-\frac{1}{\sin^2\theta}A_\theta + 2\frac{\partial A_r}{\partial \theta} - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_\phi}{\partial \phi}\right] \tag{37}$$

$$\hat{\mathbf{r}} \cdot \boldsymbol{\nabla}^2 \mathbf{A} = \nabla^2 A_r - \frac{2}{r^2}\left[A_r + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial \phi}\right] \tag{38}$$