1 Magnetostatic

For the magnetostatic case, Maxwell's equations reduce to

$$\begin{aligned} \nabla\times\mathbf{H} &= \mathbf{J} \\ \nabla\cdot\mathbf{B} &= \mathbf{0} \, . \end{aligned} \tag{1}$$

Since the divergence of $\mathbf{H}$ is zero, it can be written in terms of the magnetic vector potential $\mathbf{A}$ as

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} \, , \tag{2}$$

which leads to

$$\nabla\times\nabla\times\mathbf{A} = \mu\mathbf{J} \, . \tag{3}$$

Since $\nabla\times\nabla\times\mathbf{A} = \nabla\nabla\cdot\mathbf{A} - \nabla^2\mathbf{A}$, we have

$$\nabla\times\nabla\times\mathbf{A} = \nabla\nabla\cdot\mathbf{A} - \nabla^2\mathbf{A} = \mu\mathbf{J} \, . \tag{4}$$

The obvious choice for the divergence of $\mathbf{A}$ is

$$\nabla\cdot\mathbf{A} = 0 \, . \tag{5}$$

The electric current $\mathbf{J}$ is ϕ-symmetric, $\hat{\boldsymbol{\phi}}$-driected, and located at $(\rho', z') = (r', \theta')$. Several of the multiple methods to solve for the field are derived. If the electric current is assumed to be ϕ-symmetric, then the magnetic field is seen to be TM_ϕ and TE_r in spherical coordinates and TM_ϕ and TE_z is cylindrical coordinates. Hence, we may use a magnetic vector potential A_ϕ in both spherical and cylindrical coordinates, an electric vector potential F_r in spherical cooridnates, or an electric vector potential F_z in cylindrical coordinates to represent the fields.

1.1 Spherical Coordinate Solution

In spherical coordinates, the electric current $\mathbf{J}$ is

$$\mathbf{J}(\mathbf{r}) = J_\phi\hat{\boldsymbol{\phi}} = \frac{1}{r'}\delta(r-r')\delta(\theta-\theta')\hat{\boldsymbol{\phi}} \, . \tag{6}$$

To verify that this is the proper form for $\mathbf{J}$, we integrate over a surface S perpindicular to $\hat{\boldsymbol{\phi}}$

$$\iint_S \mathbf{J}(\mathbf{r})\cdot \mathrm{d}\mathbf{s} = \int_{r'-\Delta}^{r'+\Delta}\int_{\theta'-\Delta}^{\theta'+\Delta} \frac{1}{r'}\delta(r-r')\delta(\theta-\theta')\hat{\boldsymbol{\phi}}\cdot r\hat{\boldsymbol{\phi}}\mathrm{d}\theta\mathrm{d}r = 1 \, . \tag{7}$$

1.1.1 Magnetic Vector Potential

Since the field is TM_ϕ and $\mathbf{J}$ is $\hat{\boldsymbol{\phi}}$-directed, the solution can be written in terms of the magnetic vector potential $\mathbf{A} = A_\phi\hat{\boldsymbol{\phi}}$. The differential equation satisfied by A_ϕ is

$$\nabla^2(A_\phi\hat{\boldsymbol{\phi}}) = -\mu\mathbf{J} \tag{8}$$

since the divergence of $\mathbf{A}$ is identically zero. The ϕ-component of the vector ∇^2 operator is

$$\hat{\boldsymbol{\phi}}\cdot\nabla^2\mathbf{A} = \nabla^2 A_\phi + \frac{1}{r^2\sin\theta}\left[-\frac{1}{\sin\theta}A_\phi + 2\frac{\partial A_r}{\partial\phi} + 2\frac{\cos\theta}{\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right] \, . \tag{9}$$

Since the field is ϕ-symmetric, the differential equation reduces to

$$\nabla^2_{r\theta} A_\phi - \frac{1}{r^2 \sin^2\theta} A_\phi = -\mu J_\phi = -\mu \frac{1}{r'} \delta(r - r')\delta(\theta - \theta') \ . \tag{10}$$

At first glance, this does not appear to be the wave equation, but a careful inspection shows that it is. The solution may be written in terms of a sum of solutions of the homogeneous equation. If $A_\phi = R(r)\Theta(\theta)$, then the homogeneous equation is

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) - \frac{1}{\sin^2\theta} = 0 \ , \tag{11}$$

which is a homogeneous wave equation. Solutions are $r^n P_n^1(\cos\theta)$ and $r^n Q_n^1(\cos\theta)$. Since the field must be finite away from the source, the field may be represented as

$$\begin{aligned} A_\phi(r,\theta) &= \sum_{n=1}^{\infty} B_n r^n P_n^1(\cos\theta) \ \ , r < r' \\ &= \sum_{n=1}^{\infty} C_n \frac{1}{r^{n+1}} P_n^1(\cos\theta) \ \ , r > r' \end{aligned} \tag{12}$$

The unknown coefficients can be determined as follows. The potential should be continuous at $r = r'$, which can be enforced if

$$\begin{aligned} A_\phi(r,\theta) &= \sum_{n=1}^{\infty} A_n \left(\frac{r}{r'}\right)^n P_n^1(\cos\theta) \ \ , r < r' \\ &= \sum_{n=1}^{\infty} A_n \left(\frac{r'}{r}\right)^{n+1} P_n^1(\cos\theta) \ \ , r > r' \end{aligned} \tag{13}$$

The differential equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_\phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A_\phi}{\partial \theta}\right) - \frac{1}{r^2\sin^2\theta} A_\phi = -\mu \frac{1}{r'}\delta(r - r')\delta(\theta - \theta') \ . \tag{14}$$

We note that a solution ψ_n to the homogeneous wave equation must satisfy

$$\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi_n}{\partial \theta}\right) - \frac{1}{r^2\sin^2\theta}\psi_n = -n(n+1)\frac{\psi_n}{r^2} \ , \tag{15}$$

so that the differential equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_\phi}{\partial r}\right) - n(n+1)\frac{A_\phi}{r^2} = -\mu \frac{1}{r'}\delta(r - r')\delta(\theta - \theta') \ . \tag{16}$$

Next, the orthogonality relation

$$T_{nm} = \int_0^\pi \sin\theta P_n^1(\cos\theta) P_m^1(\cos\theta) \mathrm{d}\theta = \delta_{nm} \frac{2}{2n+1}\frac{(n+1)!}{(n-1)!} \tag{17}$$

is used to obtain

$$T_{nn}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_n(r)}{\partial r}\right) - n(n+1) T_{nn} A_n(r) = -\mu r^2 \frac{\sin\theta'}{r'}\delta(r - r') P_n^1(\cos\theta') \ . \tag{18}$$

by integrating the differential equation with $\int_0^\pi (\cdot)\, P_n^1(\cos\theta)\sin\theta \mathrm{d}\theta$. If we apply the so-called "jump-condition" and integrate with respect to r over $(r'-\Delta, r'+\Delta)$ and let $\Delta \to 0$, we obtain

$$\left.\frac{\partial A_n(r)}{\partial r}\right|_{r'-}^{r'+} = -\mu\frac{1}{T_{nn}}\frac{\sin\theta'}{r'}P_n^1(\cos\theta')\ . \tag{19}$$

This gives

$$-A_n\frac{2n+1}{r'} = -\mu\frac{1}{T_{nn}}\frac{\sin\theta'}{r'}P_n^1(\cos\theta')\ , \tag{20}$$

or

$$A_n = \frac{\mu}{2}\frac{(n-1)!}{(n+1)!}\sin\theta' P_n^1(\cos\theta')\ . \tag{21}$$

Therefore, the solution is

$$\begin{aligned} A_\phi(r,\theta) &= \frac{\mu}{2}r'\sin\theta'\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+1)!}\frac{1}{r'}\left(\frac{r}{r'}\right)^n P_n^1(\cos\theta)P_n^1(\cos\theta')\quad ,r<r' \\ &= \frac{\mu}{2}r'\sin\theta'\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+1)!}\frac{1}{r'}\left(\frac{r'}{r}\right)^{n+1} P_n^1(\cos\theta)P_n^1(\cos\theta')\quad ,r>r'\ , \end{aligned} \tag{22}$$

which is written compactly as

$$A_\phi(r,\theta) = \frac{\mu}{2}\frac{r'}{r_>}\sin\theta'\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+1)!}\left(\frac{r_<}{r_>}\right)^n P_n^1(\cos\theta)P_n^1(\cos\theta')\ . \tag{23}$$