Combinatorics

Combinations vs. Permutations

Order does not matter, it is a combination.

If order does matter, it is a permutation.

Example:

Let \( n \) be the number of possible outcomes.

For \( n = 4, 2, 3 \), let \( r \) be the number of elements.

There are \( \binom{n}{r} \) permutations of \( n = 3 \) and \( r = 3 \).

For combinations:

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

For permutations:

\[
\pi(n) = n!
\]

For permutations of 123:

\[
\begin{align*}
123 & \\
132 & \\
213 & \\
231 & \\
312 & \\
321 & \\
\end{align*}
\]

\( \sum_{3!} = 6 \) permutations
Sampling with replacement

Assume $n$ possible outcomes such as number balls 1 through $n$

We sample $r$ balls and replace the ball each time we sample

The number of permutations $n^r$

Example: Let $n = 2$, $r = 3$

We predict $2^3 = 8$ permutations

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<thead>
<tr>
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<tr>
<td>4</td>
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<td>3</td>
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<td>7</td>
<td>222</td>
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<td>8</td>
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Sampling without replacement

First sample, \( n \) balls are available

Second sample, \( n-1 \) balls

etc.

Number of possible ordered samples is

\[ n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!} \]

where \( r! = 1 \cdot 2 \cdot 3 \cdots \cdot r \)

and \( 0! = 1 \)

Example: Consider a group of 4 numbered balls.

What different permutations or combinations can you select 2 balls?

\( n = 4 \quad r = 2 \)
Since \( r = 2 \) we can use a matrix to visualize this set.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 1,2 & 1,3 & 1,4 \\
2 & 2,1 & 2,2 & 2,3 & 2,4 \\
3 & 3,1 & 3,2 & 3,3 & 3,4 \\
4 & 4,1 & 4,2 & 4,3 & 4,4 \\
\end{array}
\]

6 combinations

So there are \( 16 - 4 = 12 \) permutations where \( \frac{16}{6} = n^2 \).

For combinations we have

\[
\frac{n^2 - n}{2} = 6 = \frac{n!}{n! (n-1)!}
\]

\[
= \frac{4!}{2! \cdot 2!} = 6
\]
Binomial Coefficient

For a combination without replacement, we have the binomial coefficient.

\[ C^n_r = \binom{n}{r} = \frac{n!}{(n-r)! r!} \]

unique combinations

Example: \( n = 4, r = 2 \Rightarrow \frac{4!}{2!2!} = 6 \)
Bermoulli Trials and Binomial Probability

Outcome is Binary

Let \( A_k \) represent the probability of \( k \) 1's in \( n \) trials where the outcomes are either "0" or "1".

Then \( P(A_k) = \binom{n}{k} p^k q^{n-k} \)

where \( P(A_k = 1) = p \) and \( P(A_k = 0) = q = 1 - p \)

Ex: Given an error correction code with 1 bit error correction. The number of information bits is 8 (not counting correction bits).

It takes at least 2 bit errors to corrupt a word.

Let \( E \equiv \) word has 0 or 1 bit error \( \equiv \) word is invalid.

\[ p = \text{prob. of a bit error} = 0.1 \]

\[ q = \text{prob. of a correct bit} = 1 - p = 0.9 \]
\[ P(E) = P(0 \text{ or } 1 \text{ bit error}) \]
\[ = \sum_{k=0}^{1} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{1} \binom{8}{k} (0.1)^k (0.9)^{8-k} = 0.81 \]

\[ \binom{8}{0} = \frac{8!}{8!0!} = 1 \]
\[ \binom{8}{1} = \frac{8!}{7!1!} = 8 \]

\[ 1 \cdot 1^0 (0.9)^8 + 8 \cdot 1^1 (0.9)^7 \]