Random Variables (r.v.)

A r.v. is a real function of elements of a sample space $S$

Ex: $\tilde{X}(s)$ is a random variable as a function of $S$

$\tilde{X} = \tilde{X}(s) = s^2$

let $0 < s < 12$
then $0 < X < 144$

Sampling 2:1 mapping or non-unique mapping
Prob. Measure of r.v. \( \xi \)

Like random events there are axioms to define a r.v. 

\[ P(\{ \xi \leq x \}) \]  

the prob. that the r.v. \( \xi \) is less than or equal to a deterministic value \( x \)

\[ P(\{ \xi = x \}) = P(\{ \xi = -x \}) = 0 \]

We already defined a prob. measure, \( \mu(Q) \).

\[ 0 \leq P(\{ \xi \leq x \}) \leq 1 \]

Discrete and continuous r.v.

- Discrete r.v.'s have only discrete values
- Continuous r.v.'s have continuous values
- They can be mixed
Cumulative probability distribution function (cdf)

\[ F_X(x) = P(X \leq x) \]

Properties

1. \( F_X(-\infty) = 0, \ F_X(\infty) = 1 \)
2. \( 0 \leq F_X(x) \leq 1, \ F_X(x_1) \leq F_X(x_2) \) if \( x_1 < x_2 \)
3. Add other properties
4. \( F_X(x^+) = F_X(x) \)
5. \( F_X(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) \)
Probability density function (pdf)

\[ f_{\tilde{X}}(x) = \frac{d}{dx} F_{\tilde{X}}(x) \]

In a "less rigorous" way, the pdf represents the density of \( \tilde{X} \sim X \). For continuous r.v.'s, it is only a conceptual aid since \( P(x \leq \tilde{X} \leq x) = 0 \).

Properties of a pdf:

1. \( 0 \leq f_{\tilde{X}}(x) \) \( \forall x \)

2. \( \int_{-\infty}^{\infty} f_{\tilde{X}}(x)dx = 1 \)

3. \( F_{\tilde{X}}(x) = \int_{-\infty}^{x} f_{\tilde{X}}(t)dt \)

4. \( P(x_1 < \tilde{X} \leq x_2) = \int_{x_1}^{x_2} f_{\tilde{X}}(x)dx \)
The Gaussian density is the most important of all densities and it occurs in nearly all areas of science and engineering. This importance stems from its accurate description of many practical and significant real-world quantities, especially when such quantities are the result of many small independent random effects acting to create the quantities of interest. For example, the voltage across a resistor at the output of an amplifier can be random (a noise voltage) due to a random current that is the result of many contributions from other random currents at various places within the amplifier. Random thermal action can be random (a noise voltage) across a resistor at the output of an amplifier to create the quantities of interest. For example, the voltage across a resistor at the output of an amplifier can be random (a noise voltage) due to a random current that is the result of many contributions from other random currents at various places within the amplifier. The Gaussian density is called a Gaussian because the random variable representing the noise voltage has a normal distribution. The random variables of the various currents, this type of noise is

\[ x = x_d + x_n \]

where \( x_d \) is a fixed quantity and \( x_n \) is a random variable. The spread of a Gaussian distribution is determined by the standard deviation \( \sigma \).

**Figure 2.4-1.** The spread of a Gaussian distribution is determined by the standard deviation \( \sigma \).
The Gaussian r.v. 
\( \tilde{X} \) is defined by its pdf

\[
f_{\tilde{X}}(x) = \frac{1}{\sqrt{2\pi\sigma^2_{x}}} e^{-\frac{(x-\mu_{x})^2}{2\sigma^2_{x}}}
\]

and its cdf is

\[
F_{\tilde{X}}(x) = \int_{-\infty}^{x} f_{\tilde{X}}(\lambda) d\lambda = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2_{x}}} e^{-\frac{(\lambda-\mu_{x})^2}{2\sigma^2_{x}}} d\lambda
\]

Notation: \( \tilde{X} \sim N(\mu, \sigma^2) \)
The normalized case is where
\[ M = 0 \quad \text{and} \quad \sigma^2 = 1 \]

For negative \( x \) values we use \( F(-x) = 1 - F(x) \)

Note that \( F(0) = 0.5 \) for the normalized case.

We make a variable change \( \lambda = \frac{x-M}{\sigma} \)

so \( F_X \left( \frac{x-M}{\sigma} \right) = F_X(x) \) or \( F_X(\lambda) = F_X(x) \)

\[
\underline{\text{normalized}} \quad \underline{\text{true cdf}}
\]
EX: Given a Gaussian r.v. with $\mu = 3$ and $\sigma = 2$

Find $P(\bar{X} \leq 5.5) = ?$ or $X = 5.5$

$\chi = \frac{5.5 - 3}{2} = \frac{2.5}{2} = 1.25$

$Q(y) = 0.1056$

$Q(y) = \int_{y}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 - \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$P(\bar{X} \leq 5.5) = 1 - Q(y) = 1 - 0.1056$
PDFs

**Uniform PDF**

\[ x \sim U(a, b) \]

\[ \text{is a r.v. having a uniform distribution between } a \text{ and } b. \]

The pdf is

\[ f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases} \]

\[ \text{Rect} \left( \frac{x-a}{b-a} \right) \]

Test:

\[ \int_{-\infty}^{\infty} f(x) \, dx = \int_{a}^{b} \frac{1}{b-a} \, dx = \frac{1}{b-a} \left[ x \right]_{a}^{b} = 1 \]
Ex: Model a random bit value with a uniform r.v.

For a "1" bit \( p(1) = p \), and "0" \( p(0) = q \).

The model requires a threshold \( \eta \) and a uniformly distributed random number generator.

Let \( \tilde{x} \sim U(0,1) \) be approximated by MATLAB pseudo-random number generator RAND().

\[
\begin{array}{c}
\text{if } \tilde{x} < \eta \text{ then "0" bit else a "1" bit.} \\
\text{From the cdf we know } F_{\tilde{x}}(x) = P(\tilde{x} < x) = \int_{-\infty}^{x} f_{\tilde{x}}(t) dt \\
\text{For our model } p(\text{"0"}) = F_{\tilde{x}}(\eta) = \int_{0}^{\eta} f_{\tilde{x}}(x) dx = \int_{0}^{\eta} dx = \eta \\
\text{when } 0 \leq \eta \leq 1 \quad p_{11} = \int_{0}^{\frac{1}{2}} f_{\tilde{x}}(x) dx = \int_{0}^{\frac{1}{2}} dx = \frac{1}{2} = \eta \quad p_{01} = \int_{\frac{1}{2}}^{1} f_{\tilde{x}}(x) dx = \int_{\frac{1}{2}}^{1} dx = 1 - \frac{1}{2} = \frac{1}{2} = \eta \\
\text{for equally likely}
\end{array}
\]
Different Distributions

Exponential: \( f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases} \)

\[
\begin{align*}
\lambda & \downarrow \\
\uparrow & x \\
\end{align*}
\]

\( F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases} \)

Gamma Distribution

\( \alpha \) and \( \beta \) are parameters and result in a variety of pdf shapes.

\( f_X(x) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^\alpha} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases} \)

where \( \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx \)
If $\alpha$ is an integer

$$\Gamma(n) = (n-1) \Gamma(n-1) = (n-1)!$$

so for $\alpha = n$, $\beta = \frac{1}{\lambda}$

$$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

$\alpha = 0.5, \beta = 1$

$\alpha = 4, \beta = 2$

$\alpha = 10, \beta = 2$

$\beta = 0.5$

$\beta = 2$

$\alpha = 2$
Rayleigh Distribution

\[ f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} & x \geq 0 \\ 0 & \text{else} \end{cases} \]

\[ F_X(x) = \begin{cases} 1 - e^{-x^2/2\sigma^2} & x \geq 0 \\ 0 & x < 0 \end{cases} \]

Chi-Square Distribution

\[ x \text{ is chi-square } \chi^2(n) \text{ with } n \text{ degrees of freedom} \]

\[ f_X(x) = \begin{cases} \frac{x^{n/2-1} e^{-x/2}}{2^{n/2} \Gamma(n/2)} & x \geq 0 \\ 0 & \text{else} \end{cases} \]