**Linear System**

\[ X(t) \xrightarrow{h(t)} Y(t) \]

\[ Y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda \]

\[ Y(\mathcal{F}) = X(\mathcal{F}) H(\mathcal{F}) \quad \text{where} \quad H(\mathcal{F}) = \frac{Y(\mathcal{F})}{X(\mathcal{F})} \quad \text{Transfer function of filter} \]

\[ h(t) \quad \text{impulse response of filter} \]

let \( x(t) = \delta(t) \)

\[ Y(t) = \int_{-\infty}^{\infty} \delta(\lambda) h(t-\lambda) d\lambda = h(t) \]
Linearity and superposition

Let \( g(x(t)) = y(t) \)

\[ y(t) = g(x_1(t), x_2(t)) \]

If \( g(.) \) is linear then

\[ y(t) = \alpha_1 g(x_1(t)) + \alpha_2 g(x_2(t)) \]

\[ = y_1(t) + y_2(t) \]

Stability

A linear system is bounded input bounded output (BIBO) stable if every bounded input results in a bounded output.

We have BIBO stability if \( \int_0^\infty |h(t)|\,dt < \infty \)
Causality
A system is causal if it does not anticipate the input. For time-invariant causal system

\[
\text{Note: } h(t) = 0, \quad t < 0
\]

in convolution \( h(t - t) \)

\[
\text{x(t) \quad \lambda}
\]

\[
\lambda
\]
Symmetry Properties of $H(f)$

Let $H(f) = |H(f)| \exp \left( j \angle H(f) \right)$

For a real-time function $h(t)$

$|H(f)| = |H(-f)| \quad \text{(even)}$

and

$\angle H(f) = -\angle H(-f) \quad \text{(odd)}$

Power Transfer Function

Consider an energy signal (finite length) $y(t)$ as a response $y(t) = x(t) * h(t)$

The auto-correlation is $R_{yy}(\tau) = \int_{-\infty}^{\infty} y(x) \* y(x-\tau) \, dx$
The Power Spectral Density is
\[ P_{yy}(\tau) = \frac{1}{\pi} \frac{1}{R_{yy}(\tau)} = Y(\tau) Y^*(\tau) = |Y(\tau)|^2 \]
\[ P_{yy}(\tau) = \frac{X(\tau) H(\tau) X^*(\tau) H^*(\tau)}{P_{xx}(\tau)} = \frac{|X(\tau)|^2}{|H(\tau)|^2} \]
\[ P_{yy}(\tau) = \frac{P_{yy}(\tau)}{P_{xx}(\tau)} \]

Power Transfer Function = \[ \frac{|H(\tau)|^2}{P_{xx}(\tau)} \]

Finding Power Transfer of a Power Signal
\[ P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} y^2(t) \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} y^2(t) \, dt \]
Using Parseval's Theorem
\[ P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{1}{|H(\tau)|^2} \, df \]
\[
\rho = \sum_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{T} |Y_T(f)|^2 \text{ d}f
\]

PSD of \( y \) or \( P_{yy}(f) \)

\[
P_{yy}(f) = \lim_{T \to \infty} \frac{1}{T} |Y_T(f)|^2
\]

and \( |Y_T(f)|^2 = |X_T(f)|^2 / |H(f)|^2 \)

\[
P_{yy}(f) = |H(f)|^2 \lim_{T \to \infty} \frac{1}{T} \frac{|X_T(f)|^2}{|H(f)|^2} = P_{xx}(f)
\]

Power Transfer Function

\[
|H(f)|^2 = \frac{P_{yy}(f)}{P_{xx}(f)}
\]
Ex: PTF of a simple RC circuit

\[ X(t) \xrightarrow{R} g(t) \xrightarrow{C} \quad \Rightarrow \quad \frac{Z_1 = R}{Z_c} = \frac{1}{\frac{1}{2\pi f} + RC} \]

\[ H(f) = \frac{Y(f)}{X(f)} = \frac{Z_2}{Z_1 + Z_2} = \frac{1}{\frac{1}{2\pi f} + RC} = \frac{1}{1 + (2\pi f RC)^2} \]

Power Transfer Function
Group and Phase Delay

Group delay is defined by

\[ T_g(t) = \frac{-1}{2\pi} \frac{d\phi(t)}{df} \]

Where we have \[ Y(t) = |Y(f)| e^{j\phi(t)} \]
Hilbert Transform:
\[ \tilde{m}(t) = \frac{1}{\pi t} = \frac{1}{\pi} \cdot \frac{1}{t} \]

where \( h(t) = \frac{1}{\pi t} \Rightarrow H(\omega) = \frac{1}{\omega} \).

Example:
\[ m(t) = \frac{1}{\sin(t)} \]

then \( \tilde{m}(t) = \frac{1}{\sin(t)} \)