Types of Stationarity Random Processes (r.p) 2-22-05
Rotate the constellation

\[ L = 1 - \sin \frac{\pi}{8} \]
\[ P(A) = P(X_1 > \sin \frac{\pi}{8}) \]

where \( \theta = \frac{\pi}{8} \)

so \[ P(A) = Q \left( \frac{\sin \frac{\pi}{8}}{O_{X_1}} \right) \]

since \( P(A) = P(B) \) then \( P(\hat{X} \neq \bar{X}) \leq 2 P(A) = 2 Q(\sin \frac{\pi}{8}) \)

see M-ary Phase Shift Keying for other analysis.
R. P. Stationarity

Stationarity

Strict Sense Stationarity (SSS)
Wide Sense Stationarity (WSS)
Cyclic Stationarity
Ergodicity

SSS
$\tilde{X}(t)$ is SSS if all statistics (cdfs and pdfs) of
$\tilde{X}_1 = \tilde{X}(t_1 + \tau)$, $\tilde{X}_2 = \tilde{X}(t_2 + \tau)$, ..., $\tilde{X}_k = \tilde{X}(t_k + \tau)$
are identical for all $\tau$

That is $f_{\tilde{X}_1}(x; t_1 + \tau) = f_{\tilde{X}_2}(x; t_2 + \tau)$
We usually assume a weaker definition

**WSS:** Wide Sense Stationary

\[ E \{ X(t) \} = m_x \]

and

\[ E \{ X(t) \cdot X(t+\tau) \} = R_{xx}(\tau) \]

Thus **WSS** assumes a constant mean and an autocorrelation dependent only on time difference \( \tau = t_2 - t_1 \).

Many signals are cyclic stationary in that

\[ f_{\tilde{X}}(X; t) = f_{\tilde{X}}(X; t+T) \]

where \( T \) is the period of the stationary. Within a period \( X(t) \) may be nonstationary.
Examples of Stationarity

Stationary white Gaussian Noise (WGN)

Definition

\[ \tilde{\omega}(t) \sim N(0, \sigma^2) \quad \forall t \]

and

\[ R_{\omega \omega}(t_1, t_2) = R_{\omega \omega}(\tau) = e^{-\frac{\tau^2}{2\sigma^2}} \]

where \( \tau = t_2 - t_1 \)

Stationary WGN satisfies S5S and W5S

A common use of WGN is in the Additive WGN (AWGN) model

\[ \hat{S}(t) = X(t) + \tilde{\omega}(t) \]

deterministic signal
If \( x(t) = A \) then \( \tilde{s}(t) = A + \tilde{\omega}(t) \)

and \( E \{ \tilde{s}(t) \} = A \)

and \( R_{ss}(\tau) = E \{ \tilde{s}(t) \tilde{s}(t+\tau) \} = E \{ (A + \tilde{\omega}(t))(A + \tilde{\omega}(t+\tau)) \} \)

\[ R_{ss}(\tau) = A^2 + \tilde{\omega}(t)A + A\tilde{\omega}(t+\tau) + \tilde{\omega}(t)\tilde{\omega}(t+\tau) \]

\( \therefore \tilde{s}(t) \text{ is WSS} \)

Almost WSS (\( \text{AWGWN} \)) with AWGWN

If \( \tilde{s}(t) = x(t) + \tilde{\omega}(t) \) and \( x(t) \) is not a constant then

\[ E \{ \tilde{s}(t) \} = x(t) \]

\[ E \{ \tilde{s}(t) \tilde{s}(t+\tau) \} = E \{ (x(t) + \tilde{\omega}(t))(x(t+\tau) + \tilde{\omega}(t+\tau)) \} \]

\[ = E \{ x(t)x(t+\tau) + \tilde{\omega}(t)x(t+\tau) + \tilde{\omega}(t+\tau)x(t) + \tilde{\omega}(t)\tilde{\omega}(t+\tau) \} \]
\[ R_{xx}(t, t+\tau) = E \{ x(t)x(t+\tau)^2 \} + R_{\omega \omega}(\tau) \]

\[ = x(t)x(t+\tau) + R_{\omega \omega}(\tau) \]

If \( x(t)x(t+\tau) \) is deterministically stationary, i.e.
\[ A_{xx}(\tau) = x(t)x(t+\tau) \]

then
\[ R_{xx}(\tau) = A_{xx}(\tau) + R_{\omega \omega}(\tau) \]

\[ \hat{S}(\tau) \] is "almost WSS" since the mean value varies with time.

Cyclic stationarity with AWGN

Let \( \tilde{S}(t) = x(t) + \tilde{\omega}(t) \)

Let \( x(t) = \begin{cases} 1 & \text{for } 0 < t \leq T/2 \\ 0 & \text{for } T/2 < t \leq T \\ x(t) = x(t+nT) & \forall n \in \mathbb{Z} \end{cases} \)
\[ \mathbb{E} \int_0^T \dot{\omega}(t) \, dt + \int_0^T \dot{\omega}(t) \, dt = \int_0^T \frac{1}{T} \int_{T}^{2T} \ldots \chi(t) \, dt \]

binary square wave

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Ergodic Process

The practical value of ensemble statistics comes from the property of ergodicity.

That is, the ensemble states approximate the time statistics.

Ex: Mean Ergodic

\[ \mathbb{E} \mathbb{E}_x(x(t)) = \int_{-\infty}^{\infty} x(t) f_x(x(t)) \, dx \]
\[ E \{ \hat{x}(t) \}^3 = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^3(t) \, dt \]

In general, an ergodic process has the property

\[ E \{ g(\hat{x}(t)) \}^3 = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g(x^3(t)) \, dt \]